THIRD YEAR PHYSICS COLLECTION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

Time allowed: 2 hours

B5: GENERAL RELATIVITY

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

1. The Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)dt^2 + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad (1)$$

is the spherically symmetric, static solution to the Einstein field equations for a charged, non-rotating black hole of mass M and charge Q. Here, $r_s = 2GM = 2GM/c^2$ is the usual Schwarzschild radius, and $r_Q^2 = GQ^2 = GQ^2/c^4$ is a characteristic lengthscale corresponding to the radial electric field associated with the black hole.

Show that this metric has two horizons:

$$r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right).$$

Justify that consideration of geodesics can be restricted to the $\theta = \pi/2$ plane. Then, show that geodesic motion of uncharged test-particles in (1) admits two conserved quantities

$$E = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)\dot{t}, \quad J = r^2\dot{\phi},$$

where the dot denotes differentiation with respect to some affine parameter. What are the corresponding Killing vectors K^a? Show that

$$\dot{r}^2 + V_{\text{eff}}(r) = E^2 - k^2,$$

giving expressions for the effective potential $V_{\text{eff}}(r)$ and constant k. Show that a massive particle initially at rest far away from the black hole has a minimum radius that it can reach of $Q^2/2M$. Considering null geodesics, use your expression for $V_{\text{eff}}(r)$ to show that no stable circular orbit exists for $r > r_+$. [*Hint: Recall that, for some Killing vector* K^b, $g_{ab}\dot{x}^a K^b$ *is constant along affinely parametrised geodesics.*] [13]

In the remainder of this question, we shall derive the time taken for a black hole to evaporate due to Hawking radiation. For this, we will need to know the *surface gravity* κ on a horizon, defined implicitly as

$$\nabla_a(-K^b K_b)\Big|_{\text{horizon}} = 2\kappa K_a|_{\text{horizon}}, \qquad (2)$$

where K^a is a Killing vector of the metric, and both sides of this expression are evaluated on the horizon in question.

(a) To find κ, we consider the coordinate transformation

$$dv = dt + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr.$$

Re-write the metric (1) using this coordinate transformation. What are the Killing vectors in this new coordinate system? [2]

(b) By considering an appropriate Killing vector, use (2) to show that the surface gravity at the outer and inner horizons is

$$\kappa_\pm = \pm\frac{r_+ - r_-}{2r_\pm^2}.$$

[5]

(c) The blackbody luminosity of a black hole due to Hawking radiation is given by $L = A\sigma T^4$, where A is the surface area of the black hole at the horizon, σ is the Stefan-Boltzmann constant, and $T = (\hbar c)/(2\pi k_B)\kappa$ is the Hawking temperature. Show that for $Q = 0$, the time taken for the black hole to evaporate due to Hawking radiation from the outer horizon r_+ is

$$t_\infty = \frac{256\pi^3}{3}\frac{k_B^4 G^2}{\hbar^4 c^6 \sigma}M^3 = \frac{5120\pi G^2}{\hbar c^4}M^3.$$

Find the lifetime of a solar-mass black hole; do we expect to be able to observe a black hole evaporating? [5]

2. Consider a two dimensional spacetime with invariant interval

$$\mathrm{d}s^2 = e^{2g\xi}(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2),$$

where g is a positive constant.

(a) Show that

$$E = e^{2g\xi}\dot{\eta}, \quad L = e^{2g\xi}\left(-\dot{\eta}^2 + \dot{\xi}^2\right),$$

are conserved along geodesics, where the dot denotes differentiation with respect to the affine parameter. [3]

(b) By considering an equation for $(\mathrm{d}\xi/\mathrm{d}\eta)^2$, or otherwise, show that $E^2 \geq e^{2g\xi}$ for timelike observers. Hence explain why an observer following a timelike geodesic who initially moves in the $+\xi$ direction will eventually turn around and approach $\xi = -\infty$. [5]

(c) Compute the four-velocity and four-acceleration of stationary observers in this spacetime. [5]

(d) Suppose that a stationary observer at $\xi = \xi_1$ sends a photon to another stationary observer at $\xi = \xi_2$. If these observers measure frequencies ω_1 and ω_2 respectively, show that

$$\frac{\omega_2}{\omega_1} = e^{-g(\xi_2 - \xi_1)}.$$

[*Hint: consider some vector K^a satisfying $g_{ab}K^a v^b$ = constant along a geodesic with tangent v^b.*] [6]

(e) Consider some new coordinate χ such that

$$1 + g\chi = e^{g\xi}.$$

What is the metric in (η, χ) coordinates? Re-express your result of (d) in these new coordinates, and interpret your result in terms of gravitational time dilation for $g\chi \ll 1$. For $E = 1$, to what range of χ are timelike observers confined? [6]

3. Consider a family of geodesics $x^a(\lambda, s)$, where λ is the affine parameter, and s is a parameter labelling a given geodesic. We then define the vectors

$$v^a = \frac{\partial x^a}{\partial \lambda}, \quad n^a = \frac{\partial x^a}{\partial s}.$$

Give geometrical interpretations of these vectors, and write down an expression for the geodesic equation in terms of the covariant derivative and one of these vectors. Assuming a torsion free connection, show that $\nabla_n v^a = \nabla_v n^a$, where $\nabla_n = n^b \nabla_b$ and $\nabla_v = v^b \nabla_b$. Then, prove that the vector n^a satisfies:

$$\nabla_v \nabla_v n^a = R^a{}_{bcd} v^b v^c n^d, \tag{3}$$

where the Riemann curvature tensor is given by

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{ce} \Gamma^e{}_{db} - \Gamma^a{}_{de} \Gamma^e{}_{cb}.$$

Give a physical interpretation of (3); does your answer make sense in the limit of zero curvature? [10]

We now consider the weak-gravity limit, in which the metric consists of a small perturbation on a Minkowski background:

$$g_{ab} = \eta_{ab} + h_{ab}, \quad \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1), \quad |h_{ab}| \ll 1.$$

(a) State the conditions under which the metric perturbation h_{ab} is said to be in the *tranverse-traceless gauge*. Are we always allowed to choose this gauge in our treatment of gravitational radiation? [3]

(b) Show that the Newtonian limit of (3) is given by

$$\frac{\partial^2 n^a}{\partial t^2} = \frac{1}{2} n^b \frac{\partial^2 h^a{}_b}{\partial t^2}$$

when h_{ab} is in the traceless-transverse gauge, and satisfies $h_{00} = 0$. [7]

(c) Consider a metric perturbation of the form

$$\begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} = h_\times \sin\left[\omega(z - t)\right] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with all other components being zero. Suppose that such a perturbation impinges on two particles of equal mass, initially stationary in the $z = 0$ plane at $(x, y) = \pm(a/2, 0)$. Using your result of part (a), calculate the time evolution of the separation of the two particles as a function of time, to first order in $h_\times$. [5]

4. The *Friedmann-Robertson-Walker (FRW) metric*

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right], \tag{4}$$

is a solution to Einstein's field equations over a three-dimensional manifold of constant curvature for scale factor $a(t)$. One can then show that Einstein's field equations in the presence of a cosmological constant Λ reduce to the *Friedmann equations*:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}.$$

Here, ρ and p are the energy densities and pressures respectively of an isotropic, perfect fluid. Show that the energy density ρ satisfies the continuity equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0.$$

By adopting the equation of state for a polytropic fluid $p = w\rho$, find how the density depends on the scale factor for general w. Consider the cases of pressureless matter ($w = 0$) and radiation ($w = 1/3$), and give physical explanations for the dependence of each on the scale factor. [4]

Consider a universe consisting of only matter, with a non-zero cosmological constant Λ. Show that this has a static solution for which the density and scale factor are given by

$$\rho_0 = \frac{\Lambda}{4\pi G}, \quad a_0^2 = \frac{1}{\Lambda}.$$

What value must k take in such a model? By linearising around this state of equilibrium, or otherwise, show that such a universe will be unstable to small perturbations in the scale factor. Give an example of an observation that demonstrates that we do not exist in a static universe. [8]

In the remainder of this question, we shall consider the so-called *horizon problem* of cosmology. *Recombination* refers to the time at which charged electrons and protons first become bound to form electrically neutral hydrogen atoms. At this point, photons became decoupled from the remaining matter, and began to freely stream across the Universe; we now observe these photons as the cosmic microwave background (CMB). Henceforth assume that we are in a flat universe with no cosmological constant.

(a) Write down an expression for the redshift factor z in terms of $a(t)$, the scale factor at some time t, and $a_0 = a(t_0)$, the scale factor at current time. [1]

(b) If the temperature at recombination was 3000 K, and the current temperature of the CMB is 2.72 K, estimate the redshift factor at recombination z_{rec}, assuming that the temperature T scales as $T \propto a^{-1}$. [2]

(c) Assuming a matter dominated universe, find expressions for the conformal time at present η_0 and the conformal time at recombination η_{rec} in terms of z_{rec}. [5]

(d) Calculate the ratio η_{rec}/η_0 using your answer from (b), expressing your answer in degrees. What does this ratio correspond to? Hence explain why it it is puzzling that we observe the CMB temperature to be isotropic. [5]