THIRD YEAR PHYSICS COLLECTION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

Time allowed: 2 hours

B5: GENERAL RELATIVITY

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

1. The Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) dt^2 + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad (1)$$

is the spherically symmetric, static solution to the Einstein field equations for a charged, non-rotating black hole of mass M and charge Q. Here, $r_s = 2GM = 2GM/c^2$ is the usual Schwarszchild radius, and $r_Q^2 = GQ^2 = GQ^2/c^4$ is a characteristic lengthscale corresponding to the radial electric field associated with the black hole.

Show that this metric has two horizons:

$$r_{\pm} = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right).$$

Justify that consideration of geodesics can be restricted to the $\theta = \pi/2$ plane. Then, show that geodesic motion of uncharged test-particles in (1) admits two conserved quantities

$$E = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)\dot{t}, \quad J = r^2\dot{\phi},$$

where the dot denotes differentiation with respect to some affine parameter. What are the corresponding Killing vectors K^a? Show that

$$\dot{r}^2 + V_{\text{eff}}(r) = E^2 - k^2,$$

giving expressions for the effective potential $V_{\text{eff}}(r)$ and constant k. Show that a massive particle initially at rest far away from the black hole has a minimum radius that it can reach of $Q^2/2M$. Considering null geodesics, use your expression for $V_{\text{eff}}(r)$ to show that no stable circular orbit exists for $r > r_+$. [*Hint: Recall that, for some Killing vector* K^b, $g_{ab}\dot{x}^a K^b$ *is constant along affinely parametrised geodesics.*] [13]

Solution: As with the Schwarszchild metric, the horizons of the Reissner-Nordstrom (RN) metric occur where g_{rr} diverges, giving

$$0 = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right),$$

as required. Similarly, we can also restrict our consideration of geodesics to the $\theta = \pi/2$ plane due to spherical symmetry.

Throughout the remainder of the solution, we shall define

$$f(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}$$

Using the geodesic equation for $c = 0$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{00}\dot{x}^0) = 0 \quad \Rightarrow \quad f(r)\dot{t} = E,$$

while for $c = \phi$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{\phi\phi}\dot{x}^\phi) = 0 \quad \Rightarrow \quad r^2 \sin^2\theta\,\dot{\phi} = J,$$

with the results following for $\theta = \pi/2$. As in Schwarzschild, the corresponding Killing vectors are $K^0 = (1, 0, 0, 0)$ and $K^\phi = (0, 0, 0, 1)$.

Defining $k = \mathrm{d}\tau/\mathrm{d}\lambda$, the interval gives

$$-k^2 = -f\dot{t}^2 + f^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2.$$

Using our expressions for the conserved quantities, this can be written as

$$\dot{r}^2 + \frac{J^2}{r^2}f + k^2(f-1) = E^2 - k^2 \quad \Rightarrow \quad \dot{r}^2 + V_{\text{eff}}(r) = E^2 - k^2.$$

Thus, our expression for the effective potential is

$$V_{\text{eff}}(r) = \frac{J^2}{r^2}\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) + k^2\left(-\frac{r_s}{r} + \frac{r_Q^2}{r^2}\right).$$

For a massive particle initially at rest far away from the black hole, $E = k = 1$. Considering purely radial geodesics ($J = 0$), our expression becomes

$$\dot{r}^2 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} = 0.$$

Setting $\dot{r} = 0$, we find that $r = r_Q^2/r_s = Q^2/GM$ is the minimal radius reachable by such a particle. For null geodesics ($k = 0$), the effective potential is

$$V_{\text{eff}}(r) = \frac{J^2}{r^2}\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) = J^2\frac{(r - r_+)(r - r_-)}{r^4}.$$

Clearly, V_{eff} is only negative for radii in the range $r_- < r < r_+$, meaning that any minimum that it may have occurs in this range as well; all other extrema will be maxima since $V_{\text{eff}} > 0$ for $r \to 0, \infty$. This is particularly obvious from a sketch. This means that there is no minimum for $r > r_+$, and hence no stable circular null orbits for $r > r_+$.

In the remainder of this question, we shall derive the time taken for a black hole to evaporate due to Hawking radiation. For this, we will need to know the *surface gravity* κ on a horizon, defined implicitly as

$$\nabla_a(-K^b K_b)\Big|_{\text{horizon}} = 2\kappa K_a|_{\text{horizon}}\,, \tag{2}$$

where K^a is a Killing vector of the metric, and both sides of this expression are evaluated on the horizon in question.

(a) To find κ, we consider the coordinate transformation

$$\mathrm{d}v = \mathrm{d}t + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} \mathrm{d}r.$$

Re-write the metric (1) using this coordinate transformation. What are the Killing vectors in this new coordinate system? [2]

Solution: In the new coordinate system, the RN metric becomes

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2.$$

In this new coordinate system, we have the Killing vectors $K^v = (1,0,0,0)$ and $K^\phi = (0,0,0,1)$.

(b) By considering an appropriate Killing vector, use (2) to show that the surface gravity at the outer and inner horizons is

$$\kappa_\pm = \pm\frac{r_+ - r_-}{2r_\pm^2}.$$

[5]

Solution: As an aside, κ is usually defined as

$$(K^b\nabla_b)K_a = \kappa K_a.$$

However, it is easy to show that the expression we have adopted can be written in this form using Killing's equation:

$$\nabla_a(-K^bK_b) = -2K^b\nabla_a K_b = 2K^b\nabla_b K_a = 2\kappa K_a$$

Considering $K^a = K^0$, the left-hand side of (2) can be written as

$$\nabla_a(-g_{ab}K^aK^b) = -\nabla_a g_{vv},$$

while the right-hand side can be written as

$$2\kappa K^a = 2\kappa g_{ab}K^b = 2\kappa g_{av}.$$

This means that

$$-\nabla_a g_{vv} = \kappa g_{av} = \kappa(g_{vv}, g_{rv}).$$

Noting that at $g_{vv} = 0$, $g_{rv} = 1$ at $r = r_\pm$, we thus have that

$$\kappa = -\frac{1}{2}(\nabla_v g_{vv} + \nabla_r g_{vv}) = -\frac{1}{2}\partial_r g_{vv} = \frac{1}{2}\partial_r f(r)$$

Now,

$$f(r) = \frac{(r-r_+)(r-r_-)}{r^2} \quad \Rightarrow \quad \partial_r f(r) = -\frac{2(r-r_+)(r-r_-)}{r^3} + \frac{2r - r_+ - r_-}{r^2}$$

meaning that at the horizons, we have

$$\kappa_\pm = \pm\frac{r_+ - r_-}{2r_\pm^2},$$

which is the desired result.

(c) The blackbody luminosity of a black hole due to Hawking radiation is given by $L = A\sigma T^4$, where A is the surface area of the black hole at the horizon, σ is the Stefan-Boltzmann constant, and $T = (\hbar c)/(2\pi k_B)\kappa$ is the Hawking temperature. Show that for $Q = 0$, the time taken for the black hole to evaporate due to Hawking radiation from the outer horizon r_+ is

$$t_\infty = \frac{256\pi^3}{3}\frac{k_B^4 G^2}{\hbar^4 c^6 \sigma}M^3 = \frac{5120\pi G^2}{\hbar c^4}M^3.$$

Find the lifetime of a solar-mass black hole; do we expect to be able to observe a black hole evaporating? [5]

Solution: For $Q = 0$, the RN metric (1) simply reduces to the Schwarzschild metric, and $r_+ = r_s$, so $\kappa_+ = -\kappa_- = 1/(2r_s)$. Using the fact that $A = 4\pi r_s^2$, we have that

$$L = A\sigma T^4 = 4\pi r_s^2 \sigma \left(\frac{\hbar c}{2\pi k_B}\kappa\right)^4 = \frac{\hbar^4 c^4 \sigma}{64\pi^3 k_B^4}\frac{1}{r_s^2} = \frac{\hbar^4 c^8 \sigma}{256\pi^3 k_B^4 G^2}\frac{1}{M^2}.$$

Now, by mass energy equivalence, the energy lost due to the Hawking radiation will come from the mass of the black hole, so $L = -\mathrm{d}(Mc^2)/\mathrm{d}t$. Using this in the above expression, and integrating, arrive at the desired expression. This can also be written as

$$t_\infty = \left(\frac{M}{M_\odot}\right)^3 \times 2.1 \times 10^{67} \text{ years}.$$

The lifetime of a solar-mass black hole is more than 57 orders of magnitude longer than the present age of the universe. However, this does not even take into account the fact that such a black-hole is colder than the cosmic microwave radiation in which it sits ($T \simeq 6.17 \times 10^{-8}$ K), and so is not in thermodynamic equilibrium. Indeed, any black hole with a mass greater than approximately 0.75 of the mass of the earth is colder than the CMB, and actually *increases* in mass. As the universe expands and cools the CMB, it will eventually become possible for larger black holes to evaporate due to Hawking radiation.

2. Consider a two dimensional spacetime with invariant interval

$$\mathrm{d}s^2 = e^{2g\xi}(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2),$$

where g is a positive constant.

(a) Show that

$$E = e^{2g\xi}\dot{\eta}, \quad L = e^{2g\xi}\left(-\dot{\eta}^2 + \dot{\xi}^2\right),$$

are conserved along geodesics, where the dot denotes differentiation with respect to the affine parameter. [3]

Solution: Recall the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ac}\dot{x}^a\right) = \frac{1}{2}(\partial_c g_{ab})\dot{x}^a\dot{x}^b.$$

For $c = \eta$, this gives that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{\eta\eta}\dot{\eta}) = \text{constant} \quad \Rightarrow \quad E = e^{2g\xi}\dot{\eta}.$$

The second conserved quantity follows trivially from the metric:

$$L = -\left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^2 = e^{2g\xi}\left(-\dot{\eta}^2 + \dot{\xi}^2\right).$$

(b) By considering an equation for $(\mathrm{d}\xi/\mathrm{d}\eta)^2$, or otherwise, show that $E^2 \geq e^{2g\xi}$ for timelike observers. Hence explain why an observer following a timelike geodesic who initially moves in the $+\xi$ direction will eventually turn around and approach $\xi = -\infty$. [5]

Solution: For timelike geodesics, we have that $L = 1$, meaning that

$$-1 = e^{2g\xi}\left(-\dot{\eta}^2 + \dot{\xi}^2\right) \quad \Rightarrow \quad \left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)^2 = 1 - \frac{e^{-2g\xi}}{\dot{\eta}^2} = 1 - \frac{e^{2g\xi}}{E^2}.$$

Given that the left-hand side of this expression is positive definitive, we must have that $E^2 \geq e^{2g\xi}$. If $\dot{\xi} > 0$ initially, then the above implies that ξ will increase until $E = e^{2g\xi}$, at which there is a turning point. This means that the trajectory will always eventually turn around and approach $\xi = -\infty$.

(c) Compute the four-velocity and four-acceleration of stationary observers in this spacetime. [5]

Solution: A stationary observer has four-velocity $u^a = (\dot{\eta}, 0)$. From the usual normalisation condition:

$$u^a u_a = -1 \quad \Rightarrow \quad -e^{2g\xi}\dot{\eta}^2 = -1 \quad \Rightarrow \quad u^a = (e^{-g\xi}, 0).$$

The four-acceleration is then

$$a^c = (u^b \nabla_b) u^c = u^b \left(\partial_b u^c + \Gamma^c{}_{bd} u^d \right) = u^\eta (\partial_\eta u^c + \Gamma^c{}_{\eta\eta} u^\eta) = \Gamma^c{}_{\eta\eta} (u^\eta)^2,$$

where we have used the fact that derivatives with respect to η vanish. Using the geodesic equation for $c = \xi$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{\xi\xi}\dot{\xi}) &= \frac{1}{2}\left(\partial_\xi e^{2g\xi}\right)\left(-\dot{\eta}^2 + \dot{\xi}^2\right) \\ g_{\xi\xi}\ddot{\xi} + 2ge^{2g\xi}\dot{\xi}^2 &= ge^{2g\xi}\left(-\dot{\eta}^2 + \dot{\xi}^2\right) \\ \Rightarrow \quad \ddot{\xi} + g\dot{\xi}^2 + g\dot{\eta}^2 &= 0. \end{aligned}$$

This means that the only non-zero Christoffel symbol of the form $\Gamma^c{}_{\eta\eta}$ is $\Gamma^\xi{}_{\eta\eta} = g$. Putting this all together, this means that the four-acceleration of stationary observers is given by

$$a^c = \left(0, ge^{-2g\xi}\right).$$

(d) Suppose that a stationary observer at $\xi = \xi_1$ sends a photon to another stationary observer at $\xi = \xi_2$. If these observers measure frequencies ω_1 and ω_2 respectively, show that

$$\frac{\omega_2}{\omega_1} = e^{-g(\xi_2 - \xi_1)}.$$

[*Hint: consider some vector K^a satisfying $g_{ab}K^a v^b$ = constant along a geodesic with tangent v^b.*] [6]

Solution: For a vector K^a that is Killing for the metric g_{ab}, it is easy to show that $g_{ab}K^a v^b$ = constant along a geodesic with tangent v^b. Then, we have that

$$E = e^{2g\xi}\dot{\eta} = g_{\eta\eta}K^\eta\dot{\eta} \quad \Rightarrow \quad K^\eta = 1,\ K^\xi = 0.$$

The frequency of the photon as measured by the stationary observer will be $\omega = u^a v_a$. This means that

$$\frac{\omega_2}{\omega_1} = \frac{u_2^a v_{2a}}{u_1^a v_{1a}} = \frac{e^{-g\xi_2}K^a v_{2a}}{e^{-g\xi_1}K^a v_{1a}} = e^{-g(\xi_2 - \xi_1)},$$

as required.

(e) Consider some new coordinate χ such that

$$1 + g\chi = e^{g\xi}.$$

What is the metric in (η, χ) coordinates? Re-express your result of (d) in these new coordinates, and interpret your result in terms of gravitational time dilation for $g\chi \ll 1$. For $E = 1$, to what range of χ are timelike observers confined? [6]

Solution: The metric transformation gives

$$g\mathrm{d}\chi = \mathrm{d}e^{g\xi} = ge^{g\xi}\mathrm{d}\xi \quad \Rightarrow \quad \mathrm{d}\xi = e^{-g\xi}\mathrm{d}\chi.$$

In the new coordinates, our metric then becomes

$$\mathrm{d}s^2 = e^{2g\xi}\left(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2\right) = -(1+g\chi)^2\mathrm{d}\eta^2 + \mathrm{d}\chi^2.$$

Our result of (d) can be rewritten in these coordinates as

$$\frac{\omega_2}{\omega_1} = e^{-g(\xi_2-\xi_1)} = \frac{1+g\chi_1}{1+g\chi_2} \simeq 1 - g(\chi_2 - \chi_1),$$

meaning that the red/blue-shift in the frequency is given by the difference in the gravitational potentials $\Phi = g\chi$ in this limit. Now, we can rewrite the coordinate transformation as

$$\chi = \frac{1}{g}\left(e^{g\xi} - 1\right) \leq \frac{1}{g}(|E| - 1),$$

where we have used the result of (b). This means that for $E = 1$, $\chi \leq 0$.

3. Consider a family of geodesics $x^a(\lambda, s)$, where λ is the affine parameter, and s is a parameter labelling a given geodesic. We then define the vectors

$$v^a = \frac{\partial x^a}{\partial \lambda}, \quad n^a = \frac{\partial x^a}{\partial s}.$$

Give geometrical interpretations of these vectors, and write down an expression for the geodesic equation in terms of the covariant derivative and one of these vectors. Assuming a torsion free connection, show that $\nabla_n v^a = \nabla_v n^a$, where $\nabla_n = n^b \nabla_b$ and $\nabla_v = v^b \nabla_b$. Then, prove that the vector n^a satisfies:

$$\nabla_v \nabla_v n^a = R^a{}_{bcd} v^b v^c n^d, \tag{3}$$

where the Riemann curvature tensor is given by

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{ce} \Gamma^e{}_{db} - \Gamma^a{}_{de} \Gamma^e{}_{cb}.$$

Give a physical interpretation of (3); does your answer make sense in the limit of zero curvature? [10]

Solution: Geometrically, v^a and n^a are the tangent and normal vectors for a given family of geodesics. The geodesic equation can be expressed in terms of the parallel transport condition as $(v^b \nabla_b) v^a = \nabla_v v^a = 0$. Then, we have that

$$\begin{aligned}
\nabla_n v^a - \nabla_v n^a &= n^b \nabla_b v^a - v^b \nabla_b n^a = n^b \partial_b v^a - v^b \partial_b n^a + n^b \Gamma^a{}_{bc} v^c - v^b \Gamma^a{}_{bc} n^c \\
&= \frac{\partial x^b}{\partial s} \frac{\partial}{\partial x^b} \frac{\partial x^a}{\partial \lambda} - \frac{\partial x^b}{\partial \lambda} \frac{\partial}{\partial x^b} \frac{\partial x^a}{\partial s} = \frac{\partial x^a}{\partial s \partial \lambda} - \frac{\partial x^a}{\partial \lambda \partial s} = 0,
\end{aligned}$$

as required. To prove the geodesic deviation equation, we consider

$$\begin{aligned}
\nabla_v \nabla_v n^a = \nabla_v \nabla_n v^a &= v^c \nabla_c (n^d \nabla_d v^a) = v^c n^d \nabla_c \nabla_d v^a + (v^c \nabla_c n^d)(\nabla_d v^a) \\
&= v^c n^d \nabla_c \nabla_d v^a - v^c n^d \nabla_d \nabla_c v^a + v^c n^d \nabla_d \nabla_c v^a + (n^c \nabla_c v^d)(\nabla_d v^a) \\
&= v^c n^d (\nabla_c \nabla_d - \nabla_d \nabla_c) v^a + n^d \nabla_d (v^c \nabla_c v^a) \\
&= v^c n^d R^a{}_{bcd} v^b,
\end{aligned}$$

and so the desired result follows. We have used the fact that $\nabla_n v^a = \nabla_v n^a$ in the first equality, and moving to the second line. (3) is the geodesic deviation equation, which describes the relative acceleration of neighbouring geodesics due to the curvature of spacetime, and hence the dynamical dependence of normal vector n^a on the Riemann tensor. In the limit of vanishing curvature, geodesics are straight lines, which evidently do not diverge, which makes sense.

We now consider the weak-gravity limit, in which the metric consists of a small perturbation on a Minkowski background:

$$g_{ab} = \eta_{ab} + h_{ab}, \quad \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1), \quad |h_{ab}| \ll 1.$$

(a) State the conditions under which the metric perturbation h_{ab} is said to be in the *tranverse-traceless gauge*. Are we always allowed to choose this gauge in our treatment of gravitational radiation? [3]

Solution: h_{ab} is said to be in the transverse-traceless gauge if it satisfies the harmonic gauge condition, $\partial^a h_{ab} = 0$, as well as $h_{a0} = h_{0a} = h^a{}_a = 0$. We are always allowed to adopt the tranverse-traceless gauge due to the coordinate gauge freedom that is present.

(b) Show that the Newtonian limit of (3) is given by

$$\frac{\partial^2 n^a}{\partial t^2} = \frac{1}{2} n^b \frac{\partial^2 h^a{}_b}{\partial t^2}$$

when h_{ab} is in the traceless-transverse gauge, and satisfies $h_{00} = 0$. [7]

Solution: In the Newtonian limit, the timelike components of v^b and v^c will dominate, meaning that

$$\nabla_v \nabla_v n^a \simeq R^a{}_{00d} n^d = -R^a{}_{0d0} n^d.$$

Given that we are in the weak-gravity limit, we ignore products of the connection coefficients in the Riemann tensor. Then:

$$\begin{aligned} R^a{}_{0c0} &= \partial_c \Gamma^a{}_{00} - \partial_0 \Gamma^a{}_{0c} \\ &= \partial_c \frac{1}{2} g^{ad} (\partial_0 h_{0d} + \partial_0 h_{d0} - \partial_d h_{00}) - \partial_0 \frac{1}{2} g^{ad} (\partial_c h_{0d} + \partial_0 h_{cd} - \partial_d h_{00}) \\ &= -\frac{1}{2} g^{ad} \partial_0 \partial_0 h_{cd} = -\frac{1}{2} \partial_0 \partial_0 h^a{}_c, \end{aligned}$$

where we have used the fact that $h_{0d} = 0$ in the transverse-traceless gauge, as well as assuming that $h_{00} = 0$. This means that

$$\nabla_v \nabla_v n^a = \frac{1}{2} n^b \partial_0 \partial_0 h^a{}_b.$$

Now,

$$\nabla_v \nabla_v n^a = \nabla_v \left[v^b \left(\partial_b n^a + \Gamma^a{}_{bc} n^c \right) \right] \simeq \nabla_v \left[v^b \partial_b n^a \right] \simeq \nabla_v \partial_0 n^a \simeq \partial_0^2 n^a.$$

where we have used the fact that $v^a \simeq v^0$ to leading order. Substituting these into the above, the desired result follows.

(c) Consider a metric perturbation of the form

$$\begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} = h_\times \sin\left[\omega(z - t)\right] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with all other components being zero. Suppose that such a perturbation impinges on two particles of equal mass, initially stationary in the $z = 0$ plane at $(x, y) = \pm(a/2, 0)$. Using your result of part (a), calculate the time evolution of the separation of the two particles as a function of time, to first order in $h_\times$. [5]

Solution: Due to the form of the metric, the result of part (b) becomes

$$\frac{\partial^2 n^i}{\partial t^2} = \frac{1}{2} n^j \frac{\partial^2 h^i{}_j}{\partial t^2} \simeq \frac{1}{2} n^j(0) \frac{\partial^2 h^i{}_j}{\partial t^2},$$

where we have evaluated n^j on the right-hand side to zeroth order (equal to its value at $t = 0$), as the metric perturbation $h^i{}_j$ is already first order. Integrating, we thus have that

$$n^i(t) = \frac{1}{2} n^j(0) h^i{}_j + \dot{n}^i(0)t + n^i(0).$$

Given that the particles are initially stationary, $\dot{n}^i(0) = 0$. Given the initial conditions $n^x(0) = a$, $n^y(0) = 0$, we thus find that

$$n^x(t) = a, \quad n^y(t) = \frac{1}{2} a h_\times \sin(\omega t).$$

Thus, the particles oscillate back-and-forth relative to one another along the y-axis.

4. The *Friedmann-Robertson-Walker (FRW) metric*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right], \tag{4}$$

is a solution to Einstein's field equations over a three-dimensional manifold of constant curvature for scale factor $a(t)$. One can then show that Einstein's field equations in the presence of a cosmological constant Λ reduce to the *Friedmann equations*:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}.$$

Here, ρ and p are the energy densities and pressures respectively of an isotropic, perfect fluid. Show that the energy density ρ satisfies the continuity equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0.$$

By adopting the equation of state for a polytropic fluid $p = w\rho$, find how the density depends on the scale factor for general w. Consider the cases of pressureless matter ($w = 0$) and radiation ($w = 1/3$), and give physical explanations for the dependence of each on the scale factor. [4]

Solution: Take the time derivative of the first Friedmann equation, we have that

$$2\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = \frac{8\pi G}{3}\dot{\rho} + \frac{2k}{a^3}\dot{a}.$$

Substituting the first and second Friedmann equation into this expression, it is a line of algebra to show the desired result. Letting $p = w\rho$, we find that $\rho \propto a^{-3(1+w)}$. We consider various values of w:

- Pressureless matter ($w = 0$): $\rho_m \propto a^{-3}$. This is simply the decrease in the number density of the particles/matter as the universe is expanding with the scale factor.
- Radiation ($w = 1/3$): $\rho_\gamma \propto a^{-4}$. The energy density of radiation falls off more quickly than matter. This is because the number density of photons decreases in the same way as the number density of pressureless matter, but individual photons also lose energy as a^{-1} due to cosmological redshift.

Consider a universe consisting of only matter, with a non-zero cosmological constant Λ. Show that this has a static solution for which the density and scale factor are given by

$$\rho_0 = \frac{\Lambda}{4\pi G}, \quad a_0^2 = \frac{1}{\Lambda}.$$

What value must k take in such a model? By linearising around this state of equilibrium, or otherwise, show that such a universe will be unstable to small perturbations in the scale factor. Give an example of an observation that demonstrates that we do not exist in a static universe. [8]

Solution: Given that matter is pressureless, the steady-state Friedmann equations yield

$$0 = \frac{8\pi G}{3}\rho_0 - \frac{k}{a_0^2} + \frac{\Lambda}{3}, \quad 0 = -\frac{4\pi G}{3}\rho_0 + \frac{\Lambda}{3}.$$

Solving these simultaneously yields

$$k = 4\pi G \rho_0 a_0^2, \quad \frac{k}{a_0^2} = \Lambda \quad \Rightarrow \quad \rho_0 = \frac{\Lambda}{4\pi G}, \quad a_0^2 = \frac{1}{\Lambda}.$$

Clearly, the first expression in the line above implies that $k = 1$ (closed universe), since $\rho_0, a_0 > 0$. Letting $\rho = \rho_0 + \delta\rho$, $a = a_0 + \delta a$ in the second Friedmann equation and retaining only linear terms, we have

$$\delta\ddot{a} = -\frac{4\pi G}{3}(\rho_0 + \delta\rho)(a_0 + \delta a) + \frac{\Lambda}{3}(a_0 + \delta a) = -\frac{4\pi G}{3}a_0\delta\rho.$$

From the continuity equation, we have that

$$\frac{\delta\rho}{\rho} = -3\frac{\delta a}{a} \quad \Rightarrow \quad \delta\rho = -3\frac{\rho_0}{a_0}\delta a,$$

to linear order. This means that

$$\delta\ddot{a} = -\frac{4\pi G}{3}a_0\delta\rho = 4\pi G\rho_0\delta a = \Lambda\delta a \quad \Rightarrow \quad \delta a \propto e^{\Lambda t}.$$

Hence, such a static solution is unstable to small perturbations. Such a model would display no redshift for distant objects, which is something that we actually observe in the real universe.

In the remainder of this question, we shall consider the so-called *horizon problem* of cosmology. *Recombination* refers to the time at which charged electrons and protons first become bound to form electrically neutral hydrogen atoms. At this point, photons became decoupled from the remaining matter, and began to freely stream across the Universe; we now observe these photons as the cosmic microwave background (CMB). Henceforth assume that we are in a flat universe with no cosmological constant.

(a) Write down an expression for the redshift factor z in terms of $a(t)$, the scale factor at some time t, and $a_0 = a(t_0)$, the scale factor at current time. [1]

Solution: Here, we can simply quote the result that

$$1 + z = \frac{a_0}{a(t)}.$$

(b) If the temperature at recombination was 3000 K, and the current temperature of the CMB is 2.72 K, estimate the redshift factor at recombination z_{rec}, assuming that the temperature T scales as $T \propto a^{-1}$. [2]

Solution: From the relationship between temperature and scale factor, we have that

$$\frac{T_{\rm rec}}{T_{\rm CMB}} = \frac{a_0}{a_{\rm rec}} = 1 + z_{\rm rec} \quad \Rightarrow \quad z_{\rm rec} \simeq 1100.$$

(c) Assuming a matter dominated universe, find expressions for the conformal time at present η_0 and the conformal time at recombination $\eta_{\rm rec}$ in terms of $z_{\rm rec}$. [5]

Solution: For a matter dominated universe, the first Friedmann equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = H_0^2\left(\frac{a_0}{a}\right)^3.$$

Setting $a_0 = 1$ as usual, we have that

$$\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = H_0 a^{-3/2} \quad \Rightarrow \quad \mathrm{d}t = \frac{a^{1/2}}{H_0}\mathrm{d}a.$$

Using this result to calculate the conformal time, we have

$$\eta = \int_{a_1}^{a_2} \frac{\mathrm{d}t}{a(t)} = \frac{1}{H_0}\int_{a_1}^{a_2}\frac{\mathrm{d}a}{a^{1/2}} = \frac{2}{H_0}\left(a_2^{1/2} - a_1^{1/2}\right) = \frac{2}{H_0}\left(\frac{1}{\sqrt{1+z_2}} - \frac{1}{\sqrt{1+z_1}}\right),$$

where z_1, z_2 are the redshifts at a_1, a_2 respectively. The conformal time at present corresponds to $a_1 = 0, a_2 = 1$, while the conformal time at recombination corresponds to $a_1 = 0, a_2 = a_{\rm rec}$, such that

$$\eta_0 = \frac{2}{H_0}, \quad \eta_{\rm rec} = \frac{2}{H_0}\frac{1}{\sqrt{1+z_{\rm rec}}}.$$

(d) Calculate the ratio $\eta_{\rm rec}/\eta_0$ using your answer from (b), expressing your answer in degrees. What does this ratio correspond to? Hence explain why it it is puzzling that we observe the CMB temperature to be isotropic. [5]

Solution: Using the result for z_{rec} calculated in (b), we have that

$$\theta = \frac{\eta_{\text{rec}}}{\eta_0} = \frac{1}{\sqrt{1 + z_{\text{rec}}}} \simeq 0.03 \simeq 1.7 \text{ degrees.}$$

This ratio corresponds to the angular size of patches of the CMB that should be in causal contact with one another; light rays that reach us from the origin of the universe can only be causally connected if they were within η_{rec} of one another at recombination. However, as stated, the CMB is observed to have an istropic temperature distribution, and is thus in thermal equilibrium. This means that it is not well explained by the standard explanations of the expansion of the universe. Cosmic inflation is a possible resolution to this problem.

[**FINAL PAGE**]