**THIRD YEAR PHYSICS**

**Honour School of Physics Part B: 3 and 4 Year Courses**

**Honour School of Physics and Philosophy Part B**

**Problem Sets**

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# B5: GENERAL RELATIVITY AND COSMOLOGY

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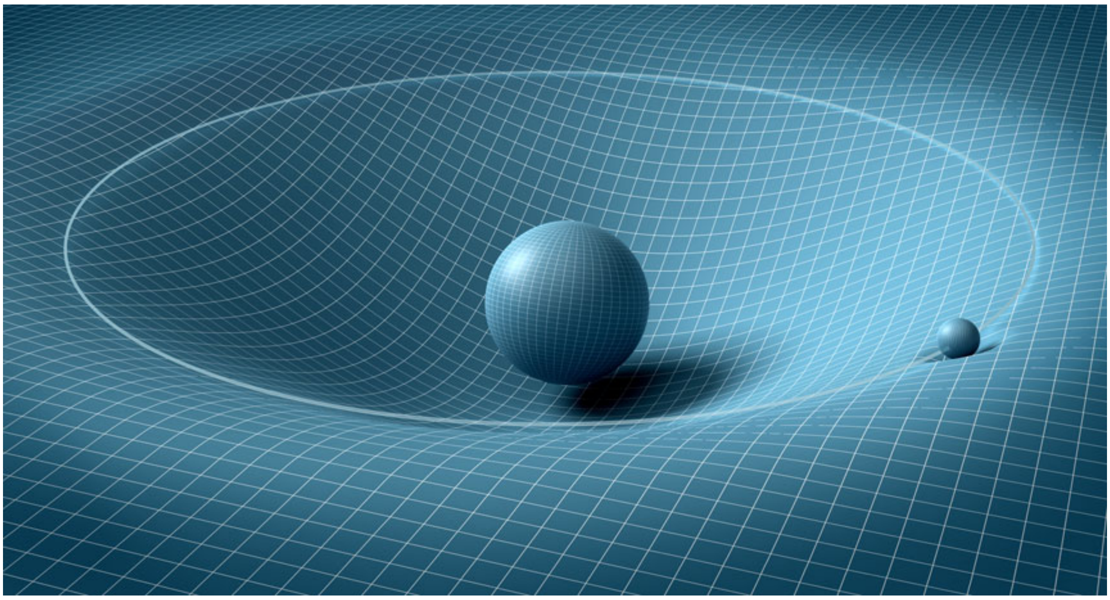
*Unless otherwise indicated, $c = 1$ throughout. $[\ldots,\ldots]$ and $\{\ldots,\ldots\}$ refers to symmetrisation and antisymmetrisation respectively. The use of a "," in a subscript denotes partial differentiation, while ";" denotes covariant differentiation with respect to the index that follows. A use of a "." above quantities corresponds to differentiation with respect to some affine parameter $\lambda$.*





# Problem Set 1: Tensors, Derivatives and Spacetime

**1.** Explain why each equation below is ambiguous or inconsistent. Provide a possible correct version of each one.

i.) $x'^a = L^{ab} x^b$

ii.) $x'^a = L^b{}_c M^c{}_d x^d$

iii.) $x'^a = L^a{}_c x^c + M^c{}_d x^d$

iv.) $\delta^a{}_b = \delta^a{}_c \delta^c{}_d$

v.) $\phi = (x^a A_a)(y^a B_a)$

**2.** We define a (contravariant) vector $v^a$ as a quantity that transforms as

$$v'^a = \Lambda^a{}_b v^b, \quad \Lambda^a{}_b \equiv \frac{\partial x'^a}{\partial x^b},$$

under a coordinate transformation $\{x\} \mapsto \{x'\}$. By demanding that the scalar $\phi = v^a \omega_a$ is invariant under coordinate transformations, derive the transformation properties of one-forms $\omega_a$. These are often referred to as *covariant vectors*.

**3.** The action of the covariant derivative on a vector $v^a$ can be written as

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c,$$

where $\Gamma^a{}_{bc}$ are known as the *Christoffel symbols* or *connection coefficients*.

i.) What is the action of the covariant derivative $\nabla_a$ on a scalar? Use this to show how $\nabla_a$ must act on a one-form $\omega_a$.

ii.) By demanding that the covariant derivative transforms as a $(1, 1)$ tensor, show that the Christoffel symbols must transform as

$$\Gamma'^b{}_{ac} = \frac{\partial x^e}{\partial x'^c} \frac{\partial x^c}{\partial x'^a} \frac{\partial x'^b}{\partial x^d} \Gamma^d{}_{ce} - \frac{\partial x^e}{\partial x'^c} \frac{\partial x^c}{\partial x'^a} \frac{\partial x'^b}{\partial x^c \partial x^e}.$$

iii.) The *Levi-civita connection* is said to be metric compatible ($\nabla_a g_{bc} = 0$) and torsion free ($\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$). By considering cyclic permutations, show that these constraints imply

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left( \partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right).$$

**4.** Consider a static spacetime with metric

$$\mathrm{d}s^2 = g_{ab} \mathrm{d}x^a \mathrm{d}x^b = g_{00} \mathrm{d}t^2 + 2 g_{0i} \mathrm{d}t \mathrm{d}x^i + g_{ij} \mathrm{d}x^i \mathrm{d}x^j,$$

where the indices $i$ and $j$ refer to spatial components of the metric. Using the normalisation condition $u^a u_a = -1$, find the four-velocity $u^a$ of a static observer in this spacetime.

Consider a spatial hypersurface such that $u_a \mathrm{d}x^a = 0$. Using this, show that the interval on such a hypersurface is given by

$$\mathrm{d}s^2_{(n-1)} = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \text{where} \quad \gamma_{ij} = g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}.$$

Furthermore, by considering $g_{ab}g^{bc} = \delta_a{}^c$, or otherwise, show that $\gamma^{ij} = g^{ik}g^{j\ell}\gamma_{k\ell} = g^{ij}$. Interpret the meaning of these results with reference to the static observer $u^a$.

**5.** Consider the metric for 3-dimensional Minkowski space $(t', r', \phi')$:

$$\mathrm{d}s^2 = -\mathrm{d}t'^2 + \mathrm{d}r'^2 + r'^2\mathrm{d}\phi'^2.$$

i.) We perform a coordinate transformation to a frame which is rotating with a constant angular velocity $\Omega$:

$$t' = t, \quad r' = r, \quad \phi' = \phi + \omega t.$$

What is the metric in the rotating frame $(t, r, \phi)$?

ii.) Consider a static observer in the rotating frame who is located at the position $(r, \phi) = (R, 0)$. How is the proper time of this observer related to the coordinate time $t$? Explain the physical significance of this result.

iii.) Compute the four-acceleration $A^a = u^b\nabla_b u^a$ for this static observer with four-velocity $u^a$. Explain the physical significance of this result.

iv.) Using your results from Question 4, compute the induced spatial metric $\gamma_{ij}$ for observers rotating with angular velocity $\omega$ in 3-dimensional Minkowski space. Using this, compute the circumference of a circle with radius $R$ as measured by these observers, and explain the physical significance of the result.

**6.** Let us start from a global inertial frame in Minkowski space $(t, x, y, z)$. Now consider the transformation to a *non-inertial* frame $(t', x', y', z')$ such that

$$t = \left(\frac{1}{g} + z'\right)\sinh(gt'), \quad x = x', \quad y = y', \quad z = \left(\frac{1}{g} + z'\right)\cosh(gt') - \frac{1}{g},$$

for some constant $g$.

i.) For $gt' \ll 1$, show that this transformation corresponds to a uniformly accelerated reference frame in Newtonian mechanics.

ii.) Plot the trajectory of the point $z' = 0$ in the inertial frame.

iii.) Show that a clock at rest at $z' = h$ runs fast compared to a clock at rest at $z' = 0$ by the factor $(1 + gh)$, as observed in the inertial frame. Use the equivalence principle to interpret this result in terms of gravitational time dilation.

iv.) What is the line element $\mathrm{d}s^2$ of a uniform gravitational field?

**7.** The energy-momentum tensor of a perfect fluid in Minkowski space is given by

$$T^{ab} = p\eta^{ab} + (p + \rho)u^a u^b,$$

where $u^a$ is the four-velocity of the fluid, and $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$. By considering an observer at rest with respect to the motion of the fluid, explain the physical meaning of $p$ and $\rho$.

The equation of motion of a perfect fluid in a local inertial frame is

$$\partial_a T^{ab} = 0. \tag{1}$$

The remainder of this question is devoted to deriving the equations of fluid mechanics from this one expression. To begin, show that the tensor

$$h^a{}_b = \delta^a{}_b + u^a u_b,$$

satisfies $h^a{}_b u^b = 0$, $h^a{}_b h^b{}_c = h^a{}_c$ and $h^a{}_a = 3$, and therefore explain why $h^a{}_b$ is a projector onto the three-dimensional hypersurfaces perpendicular to $u^a$. What is the meaning of the tensor $h_{ab} = \eta_{ac}h^c{}_b$? By projecting (1) parallel and perpendicular to the four-velocity $u^a$, show that

$$\partial_a(\rho u^a) + p\partial_a u^a = 0, \quad (p + \rho)(u^b\partial_b)u^a + h^{ab}\partial_b p = 0. \tag{2}$$

In the Newtonian limit, we approximate that $u^i \ll u^0$, $p \ll \rho$ and $|\mathbf{u}|(\partial p/\partial t) \ll |\nabla p|$. What is the physical intuition behind each of these approximations? Using these, show that (2) reduces to the familiar fluid equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \rho\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\mathbf{u} = -\nabla p.$$

# Problem Set 2: Geodesics, Curvature and Schwarzschild

**8.** Consider a curve $x^a(\lambda)$ in a metric space $g_{ab}$ parametrised by some (real) affine parameter $\lambda$. What is the condition for this curve to be time-like? By considering variations of the functional

$$S = \int \mathrm{d}\tau = \int \mathrm{d}\lambda\, \mathcal{L}(x, \dot{x}, \lambda), \quad \mathcal{L} = \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \sqrt{-g_{ab}\dot{x}^a\dot{x}^b}, \tag{3}$$

show that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ac}\dot{x}^a\right) = \frac{1}{2}(\partial_c g_{ab})\dot{x}^a\dot{x}^b, \tag{4}$$

where we recall that the dot indicates differentiation with respect to the affine parameter $\lambda$. This is the *geodesic equation.*

A geodesic curve is defined as one that parallel transports its own tangent vector. From this, show that an alternative expression for the geodesic equation is

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0. \tag{5}$$

By direct calculation, show that (4) is equivalent to (5). Lastly, by considering the total derivative of the Hamiltonian

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{x}^a}\dot{x}^a - \mathcal{L},$$

show that the Hamiltonian is conserved for the Lagrangian defined in (3). What is the condition for it to be conserved for a general Lagrangian $\mathcal{L}$?

**9.** The Lie derivative of a $(2, 0)$ tensor with respect to a vector field $x^a$ is given by

$$\mathcal{L}_x T_{ab} = (x^c\partial_c)T_{ab} + (\partial_a x^c)T_{cb} + (\partial_b x^c)T_{ac}. \tag{6}$$

Show that you can replace any $\partial_a$ with any covariant derivative $\nabla_a$ in this expression, and so argue that the Lie derivative transforms as a tensor.

Consider a vector field $K^a$ that generates a coordinate transformation $x'^a = x^a + \delta x^a = x^a + \epsilon K^a$. Show that

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}g_{cd},$$

remains invariant under this transformation if $\mathcal{L}_x g_{ab} = 0$. Show that this condition is equivalent to $\nabla_a K_b + \nabla_b K_a = 0$ for a metric compatible connection $\nabla_a$.

A vector field satisfying this property is known as a 'Killing vector', and generates an infinitesimal symmetry of the geometry defined by the metric $g_{ab}$. Show that the inner product $K_b\dot{x}^b$ is conserved along geodesics.

**10.** For (contravariant) vectors, the Riemann curvature tensor is defined as

$$[\nabla_c, \nabla_d]\, v^a = (\nabla_c \nabla_d - \nabla_d \nabla_c)\, v^a = R^a{}_{bcd} v^b. \tag{7}$$

Considering the left-hand side of this expression, show explicitly that

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}.$$

By evaluating $R^a{}_{bcd}$ in local inertial coordinates, or otherwise, show that for a metric compatible and torsion free connection $R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc}$ and $R_{a[bcd]} = R_{abcd} + R_{acdb} + R_{adbc} = 0$. Why is it sufficient to use local inertial coordinates to prove these identities?

For a metric compatible and torsion free connection, the Riemann tensor also satisfies

$$R_{ab[cd;e]} = R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0.$$

How many independent components does the Riemann tensor have in $n$ dimensions? You should find that it only has one independent component for $n = 2$. Prove that the Riemann tensor must then take the form

$$R_{abcd} = \frac{1}{2} R (g_{ac} g_{bd} - g_{ad} g_{bc}),$$

where $R$ is the Ricci scalar. Hence show that the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ vanishes in two-dimensions.

**11.** Birkoff's theorem tells us that the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2, \quad r_s = 2GM \tag{8}$$

is the unique spherically symmetric vacuum solution to the Einstein field equations in the presence of a point mass $M$. What is the physical meaning of the coordinate time $t$ in this solution? By considering the geodesic equation (4), or otherwise, show that you can always restrict attention to time-like geodesics lying in the equatorial plane, $\theta = \pi/2$.

Show that geodesics in the Schwarschild metric have two conserved quantities

$$E = \left(1 - \frac{r_s}{r}\right)\dot{t} \quad \text{and} \quad J = r^2\dot{\phi}.$$

What symmetries do these correspond to? Then, considering a change of variables $u = 1/r$, show that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + V_{\text{eff}}(u) = \frac{1}{J^2}(E^2 - k^2), \tag{9}$$

where $k = \mathrm{d}\tau/\mathrm{d}\lambda$ and we have defined some *effective potential*

$$V_{\text{eff}}(u) = u^2(1 - r_s u) - \frac{k^2}{J^2}r_s u. \tag{10}$$

Give physical interpretations of each of the terms in (10). For both timelike and null geodesics, sketch $V_{\text{eff}}$ as a function of radius $r$, finding any turning points, and give a description of the type of orbit at different values of $r$. Finally, find expressions for $E$ and $J$ in the case of a circular orbit, as well as the orbital frequency $\omega(r)$.

**12.** We define the *impact parameter* by

$$b = \frac{J}{\sqrt{E^2 - k^2}}.$$

We wish to consider when incoming geodesics will be captured by the black-hole.

i.) Show that a massless particle is captured by the black-hole if the impact parameter is smaller than a certain critical value $b < b_c$, and find an expression for the capture cross-section $\sigma = \pi b_c^2$ in terms of $M$.

ii.) Consider a massive particle that starts at $r \to \infty$ with a non-relativistic velocity $v \ll 1$ as measured by a stationary observer. Explain why $b = J/v + \mathcal{O}(v)$, and explain the physical significance of the impact parameter in this case (it may be helpful to use a diagram). Find an expression for $b_c$ in the case of the massive particle, and show that

$$\sigma = \frac{16\pi G^2 M^2}{v^2}.$$

**13.** Starting from (9), show that

$$u'' + u = \frac{3}{2}r_s u^2 + \frac{r_s}{2J^2}k^2, \quad u' = \frac{\mathrm{d}u}{\mathrm{d}\phi}.$$

i.) By considering perturbations $u = u_0 + \delta u$, $\delta u \ll u_0$ around circular orbits, find expressions for $\delta u(\phi)$ for timelike and null orbits. Using this, show that timelike circular orbits may only exist for $r > (3/2)r_s$, and that they are unstable for $r < 3r_s$. Similarly, show that null orbits are always unstable.

ii.) Mercury orbits the sun in an ellipse with semi-latus rectum of approximately $5.546 \times 10^{10}$ m. Using your results from the previous part, calculate the perihelion advance of Mercury. [*Hint: Find the correction to the orbital period due to General Relativity.*]

# Problem Set 3: Linearised Gravity

**14.** There is a class of metrics which admit coordinates such that

$$g_{ab} = \eta_{ab} + \phi\, n_a n_b,$$

with $n_a$ satisfying $\eta^{ab} n_a n_b = 0$, where $\eta_{ab} = \eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

i.) By looking for an inverse metric of the form $g^{ab} = \eta^{ab} + \psi\, n^a n^b$, show both that $n_a$ is null with respect to the metric $g_{ab}$, and that $\psi = -\phi$. [*Hint: Taking the trace may be useful here.*]

ii.) Show that $\Gamma^a{}_{bc} n^b n^c = 0$ and $\Gamma^a{}_{bc} n_a n^b = 0$. Use this to show that if $n_a$ is geodesic with respect to the Minkowski metric, $n_a \eta^{ab} \partial_b n_c = 0$, then it is also geodesic with respect to the curved metric $g_{ab}$, $n^a \nabla_a n_b = 0$.

iii.) Consider the special case for which

$$\phi = \frac{2GM}{r}, \quad n_a = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right),$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Using the results of the previous part, show that $n_a$ is geodesic with respect to this metric. Show also that $n_a \mathrm{d}x^a = dx^0 + dr$. Finally, show that the metric in question is actually the Schwarzschild solution (8). [*Hint: Look for a coordinate change of the form* $x^0 = t + \xi(r)$.]

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Throughout the remainder of this problem set, we will be working within the weak gravity limit, for which

$$g_{ab} = \eta_{ab} + h_{ab}, \quad g^{ab} = \eta^{ab} - h^{ab}, \quad |h_{ab}| \ll 1, \tag{11}$$

That is, the metric consists of a small perturbation on a Minkowski background.

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**15.** In this question, we shall fix the constant $c_1$ in Einstein's field equations

$$G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R = c_1 T^{ab}$$

where $G^{ab}$ is the Einstein tensor, $R^{ab}$ is the Ricci tensor, $R$ the Ricci scalar, and $T^{ab}$ the stress-energy tensor. Confirm that the covariant derivative of the left-hand side of this expression vanishes. What physical condition does this express?

Find an expression for the geodesic equation (5) in the weak-gravity limit (11). By comparing it with the Newtonian limit

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\nabla\Phi,$$

show that $h_{00} = -2\Phi$.

Recalling the Riemann tensor (7), show that the Ricci tensor is given by

$$R_{bd} = R^a{}_{bad} = \frac{1}{2}\left(\partial_d\partial^a h_{ab} + \partial_b\partial^a h_{da} - \partial_a\partial^a h_{bd} - \partial_b\partial_d h\right) \tag{12}$$

where $h = h^a{}_a$. Show further that the coordinate gauge transformation $h_{ab} \mapsto h_{ab} + \partial_a\xi_b + \partial_b\xi_a$ leaves (12) unchanged. Adopting the *harmonic gauge* condition

$$\partial^a \bar{h}_{ab} = \partial^a\left(h_{ab} - \frac{1}{2}\eta_{ab}h\right) = 0,$$

show that

$$(\partial^a\partial_a)\bar{h}_{bd} = -2c_1 T_{bd}. \tag{13}$$

Comparing the timelike component of (13) to the Newtonian limit $\nabla^2\Phi = 4\pi G\rho$, show that $c_1 = 8\pi G$. Restoring units, this becomes $8\pi G/c^4$, the familiar result.

**16.** We shall now consider vacuum solutions to the gravitational wave equation (13), vis.:

$$(\partial^c\partial_c)\bar{h}_{ab} = 0.$$

We seek plane-wave solutions of the form $\bar{h}_{ab} = \chi_{ab}\exp[ik_c x^c]$, for wavevector $k^a = (\omega, \mathbf{k})$ and a constant, symmetric tensor $\chi_{ab}$.

i.) Show that solutions of such form propagate at the speed of light.

ii.) Show that the wavevector $k^a$ is orthogonal to $\chi_{ab}$.

iii.) Write down the conditions for the metric perturbation to be purely spatial and traceless; perturbations satisfying these conditions are said to be in the *transverse-traceless (TT)* gauge. Show that this implies $\partial^i h_{ij} = 0$.

iv.) Finally, write down the most general form of $\chi_{ab}$ for a perturbation with wavevector $k^a = (\omega, 0, 0, \omega)$.

**17.** With some algebra, one can show from (13) that the spatial components of the metric perturbation $\bar{h}_{ij}$ at some field event $(ct, r)$ in response to a source event $(ct_s, \mathbf{r}_s)$ evolve according to the *quadrupole formula*

$$\bar{h}_{ij} = \frac{2G}{c^4}\frac{\ddot{I}_{ij}}{r}, \quad I_{ij} = \frac{1}{c^2}\int \mathrm{d}^3\mathbf{r}_s\, r_s^i r_s^j\, T^{00}(t_s, \mathbf{r}_s), \tag{14}$$

where we have restored units for the sake of clarity. Note that the time derivatives are taken with respect to the retarded time $t_s = t - r/c$ of the source event. One can also show that the gravitational luminosity of the source is given by

$$L_{\mathrm{GW}} = \frac{G}{5c^5}\dddot{J}_{ij}\dddot{J}^{ij}, \quad J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{mn}I_{mn}. \tag{15}$$

Hence, given a particular (time-dependent) distribution of matter $T^{00}$, we can find the local perturbation away from the Minkowski metric, and the resultant observed gravitational luminosity. Indeed, (15) was useful in inferring the properties of the black holes in the famous gravitational wave observation GW150914 by LIGO/Virgo.

In this question, we consider a black hole binary system merger, wherein two black-holes of masses $m_1$ and $m_2$ orbiting at $r_1$ and $r_2$ relative to the origin gradually spiral in towards one-another. We shall assume that both black holes remain on circular orbits during the merger.

i.) Assuming that the orbital motion of the bodies can be confined to the equatorial ($\theta = \pi/2$) plane, write down an expression for the time dependent mass-density of the bodies in terms of $m_1$, $m_2$, $r_1$, $r_2$ and relevant spatial coordinates. Using this, show that

$$I_{xx} = \mu r^2\cos^2\phi, \quad I_{yy} = \mu r^2\sin^2\phi, \quad I_{xy} = I_{yx} = \mu r^2\sin\phi\cos\phi,$$

where $\phi$ is the angular coordinate in the orbital plane, and $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the system.

ii.) Assuming that $\phi = \omega t$, give an expression for the orbital frequency $\omega$ in terms of $r$ and other constants. Using the quadrupole formula (15), show that the gravitational luminosity of the binary is given by

$$L_{\mathrm{GW}} = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2(m_1 + m_2)}{r^5},$$

where $r$ is the radius of motion of the centre of mass. What 'dodgy' assumption has been made in this derivation? [*Hint: your results of question 11 in the previous problem set may be useful here.*]

iii.) Using the virial theorem, or otherwise, find an expression for the total energy of the binary in terms of $r$ and other constants.

iv.) Hence, for a given initial radius $r_0$, show that the time taken for the black holes to merge is given by

$$t_{\mathrm{merge}} = \frac{5}{256}\frac{c^5}{G^3}\frac{r_0^4}{m_1 m_2(m_1 + m_2)}.$$

Does this expression scale how you would expect? Find the time taken for two black holes with equal masses $m_1 = m_2 = 60M_\odot$ initially located at one astronomical unit from one another. Is your answer reasonable?

**18.** *When two black holes of masses $m_1$ and $m_2$ collide to form a single large black hole of mass $M$, the total area of the horizon must increase.*

By considering radial, null geodesics in Scwharzschild spacetime (8), justify this statement by invoking causality. [*Hint: Think about light-cones; how are they orientated for $r < r_s$?*] Then, find an expression for an upper bound on the total energy that can be released during the merger. Find a value for this upper bound for $m_1 = m_2 = 60M_\odot$, and confirm that this is larger than the total energy emitted due to gravitational waves during the binary merger studied in question 17.

**19.** Consider two point masses $m$ located at $(\ell/2, 0, 0)$ and $(-\ell/2, 0, 0)$ respectively that are constrained to move along the $x$-axis. These are impinged upon by a gravitational wave travelling along $\hat{\mathbf{z}}$, with metric perturbation satisfying $h_{xx} = -h_{yy} = A_{xx}\cos(kz - \omega t)$. Find an expression for the proper distance of each of the masses from the origin as a function of time, to first order in the metric perturbation. Using (15), show that the time-averaged gravitational luminosity of the particle response is given by

$$\langle L_{\rm GW}\rangle_t = \frac{G}{60c^5} m^2 \omega^6 \ell^4 A_{xx}^2.$$

The energy flux due to gravitational radiation is given by

$$F^i = -\frac{c^4}{32\pi G}(\partial_i \bar{h}^{ab}\partial_t \bar{h}_{ab}).$$

The cross-section $\sigma_{\rm GW}$ for gravitational interaction is defined to be the ratio of the average luminosity to the average incoming flux. Why is this a good definition of the cross-section? Show that

$$\sigma_{\rm GW} = \frac{2\pi}{15} r_s^2 \left(\frac{\omega\ell}{c}\right)^4, \quad r_s = \frac{2Gm}{c^2}.$$

Give a physical interpretation of the factor $(\omega\ell/c)$. Evaluate this numerically for $m = 10\,\mathrm{kg}$, $\ell = 10\,\mathrm{m}$ and $\omega = 20\,\mathrm{rad\,s^{-1}}$ and compare this with the typical weak interaction cross section of $10^{-48}\,\mathrm{m}^2$. Hence justify the statement: *Gravity is the weakest force.*

# Problem Set 4: Cosmology

**20.** The *Friedmann-Robertson-Walker (FRW) metric*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left[ \frac{\mathrm{d}r^2}{1-kr^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \right], \tag{16}$$

is a solution to Einstein's field equations over a three-dimensional manifold of constant curvature.

i.) What key assumptions about the nature of our universe are used in the derivation of (16)? Give physical interpretations of the coordinate $t$, the function $a(t)$ and the constant $k$.

ii.) Consider two observers located at some fixed comoving distance $\ell$ in flat spacetime. Suppose that one observer emits a photon of wavelength $\lambda$ at time $t$, which is observed by a second observer as $\lambda_0$ at time $t_0$. Show that the cosmological redshift factor $z$ can be written as

$$z = \frac{\lambda_0}{\lambda} - 1,$$

and find an expression for $z$ in terms of $a$, $t$ and $t_0$.

iii.) Find expressions for the *Hubble constant* $H_0$ and the *deceleration parameter* $q_0$ in the expansion

$$\frac{a(t)}{a(t_0)} = 1 - H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots,$$

Then, show that for small $z$ the comoving distance $\ell$ can be expressed as

$$\ell = \frac{1}{H_0}\left[ z - \frac{1}{2} z^2 (1+q_0) + \dots \right].$$

**21.** Consider an isotropic metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j.$$

Why are there only four non-zero components of the Ricci tensor $R_{ab}$? Show explicitly that

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2k)\gamma_{ij},$$

in the case of the FRW metric (16). Here, the dot denotes differentiation with respect to the coordinate time $t$. Write down the stress-energy tensor for a perfect fluid with no overall velocity. Hence, show that Einstein's field equations in the presence of a cosmological constant $\Lambda$

$$G_{ab} = 8\pi G T_{ab} - \Lambda g_{ab},$$

reduce to the *Friedmann equations*:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{17}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{18}$$

**22.** By considering (17) and (18), show that the mass density $\rho$ satisfies the continuity equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0.$$

Show that this equation is also a consequence of stress-energy conservation. By adopting the equation of state for a polytropic fluid $p = w\rho$, find how the density depends on the scale factor for general $w$. Consider the cases of pressureless matter ($w = 0$), radiation ($w = 1/3$) and vacuum energy ($w = -1$), and give physical explanations for the dependence of each on the scale factor.

We define the *critical density* $\rho_c$ to be the density for which $k = 0$. Find an expression for $\rho_c$ in terms of the Hubble parameter $H$. Show that the first Friedmann equation (17) can be written as

$$\left(\frac{H}{H_0}\right)^2 = \left[\Omega_\gamma \left(\frac{a_0}{a}\right)^4 + \Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_k \left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda\right], \tag{19}$$

where $a_0 = a(t_0)$ is the scale factor at the current time. Give expressions for the *density ratios* $\Omega_\gamma$, $\Omega_m$, $\Omega_k$ and $\Omega_\Lambda$. Lastly, show that the second Freidmann equation (18) evaluated at the current time can be written as

$$q_0 = \frac{1}{2}\sum_i (1 + 3w_i)\Omega_i.$$

In a universe consisting of only vacuum energy, is the expansion of the universe accelerating or decelerating?

**23.** By introducing the *conformal time* $\mathrm{d}\eta = \mathrm{d}t/a$, show that the *FRW* metric (16) for $k = 0$ can be written as

$$\mathrm{d}s^2 = a(t)^2\left(-\mathrm{d}\eta^2 + dx^2 + dy^2 + dz^2\right).$$

The metric is said to be *conformally flat* on some subset of the overall space. Given that $a^2 > 0$, what is the condition for two events to be connected by a null geodesic in FRW spacetime?

Consider a universe containing pressureless matter and radiation. Show that such a cosmological model has a past, but not a future, horizon. If $a(t_0) = 1$, show that the conformal time at present is given by

$$\eta_0 = \int_0^{t_0} d\eta = 2\sqrt{\frac{8\pi G}{3\rho_m}}\left(\sqrt{1 + a_{\mathrm{eq}}} - \sqrt{a_{\mathrm{eq}}}\right),$$

and give an expression for $a_{\mathrm{eq}}$. What is it's physical interpretation?

**24.** Consider a generalised Minkowski space:

$$ds^2 = \eta_{ab} d\xi^a d\xi^b = -d\xi_0^2 + \sum_{i=1}^{n} d\xi_i^2.$$

*de-Sitter spacetime* is the maximally symmetric sub-manifold described by the constraint that

$$-\xi_0^2 + \sum_{i=1}^{n} \xi_i^2 = \alpha^2. \tag{20}$$

The Riemann tensor for such a space is given by

$$R_{\text{abcd}} = \frac{1}{\alpha^2}(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

By considering Einstein's field equations in a vacuum with a non-zero cosmological constant $\Lambda$, find a relationship between $\alpha$ and $\Lambda$ in $n$ dimensions.

Consider the parametrisation

$$\xi_0 = \sqrt{\alpha^2 - r^2}\sinh\left(\frac{t}{\alpha}\right), \quad \xi_1 = \sqrt{\alpha^2 - r^2}\cosh\left(\frac{t}{\alpha}\right),$$
$$\xi_2 = r\cos\theta, \quad \xi_3 = r\sin\theta\cos\phi, \quad \xi_4 = r\sin\theta\sin\phi.$$

Show that this satisfies the constraint (20), and that this gives rise to the interval

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

for $n = 4$. An *event horizon* is a hypersurface in spacetime that can only be crossed in one direction. Does this de-Sitter spacetime have such a horizon? Illustrate your answer using light-cone diagrams, distinguishing between the cases of $\Lambda > 0$ and $\Lambda < 0$.

**25.** The angle $\Delta\theta$ subtended by some object of size $d$ is given by $\Delta\theta = d/D_A$, where $D_A$ is the *angular distance* of the object. By considering the components of the FRW metric (16), find an expression for $D_A$ in terms of the radial coordinate distance to the object and the redshift $z$.

The *luminosity distance* $D_L$ is defined such that the flux $F$ of a body of luminosity $L$ are related by $F = L/(4\pi D_L)^2$, and is related to the angular distance by $D_L = (1+z)^2 D_A$. Show that the flux per unit area $B$ of a body of size $d$ and luminosity $L$ is given by

$$B = \frac{L}{\pi^2 d^2}\frac{1}{(1+z)^4}.$$

Why is it so hard to observe old stars and galaxies?