THIRD YEAR PHYSICS

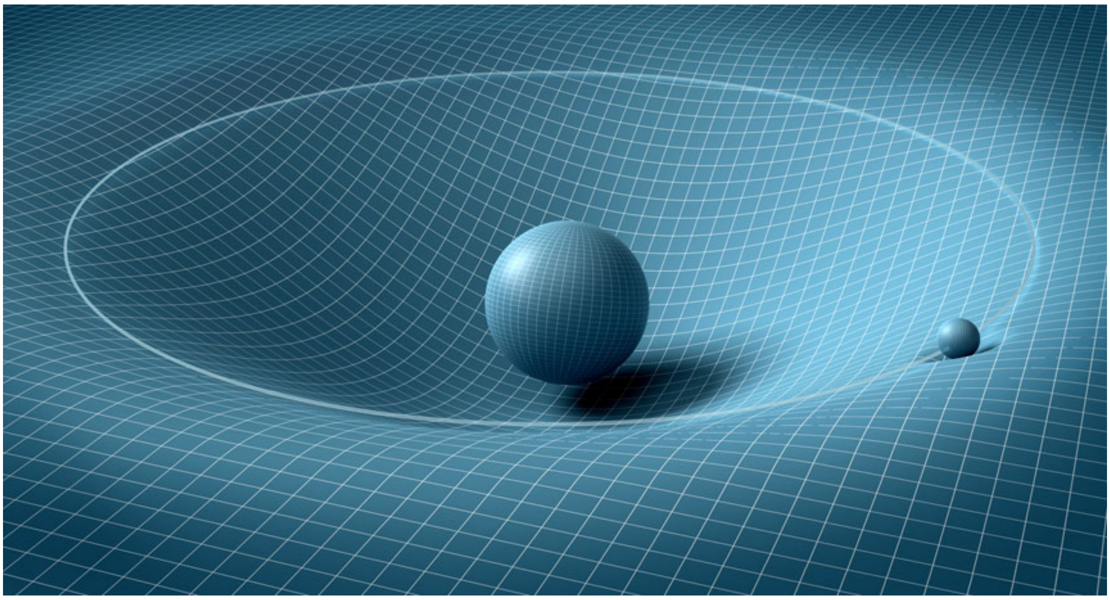
Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

Problem Sets

B5: GENERAL RELATIVITY AND COSMOLOGY

Unless otherwise indicated, $c = 1$ throughout. $[\ldots,\ldots]$ and $\{\ldots,\ldots\}$ refers to symmetrisation and antisymmetrisation respectively. The use of a "," in a subscript denotes partial differentiation, while ";" denotes covariant differentiation with respect to the index that follows. A use of a "." above quantities corresponds to differentiation with respect to some affine parameter λ.



Problem Set 1: Tensors, Derivatives and Spacetime

1. Explain why each equation below is ambiguous or inconsistent. Provide a possible correct version of each one.

i.) $x'^a = L^{ab}x^b$

Solution: There are too many free indices. Possible correction: $x'^a = L^a{}_b x^b$.

ii.) $x'^a = L^b{}_c M^c{}_d x^d$

Solution: The free index has been mislabelled. Possible correction: $x'^b = L^b{}_c M^c{}_d x^d$.

iii.) $x'^a = L^a{}_c x^c + M^c{}_d x^d$

Solution: c has already been summed over, and so cannot be used as a free index. Possible correction: $x'^a = L^a{}_c x^c + M^a{}_d x^d$.

iv.) $\delta^a{}_b = \delta^a{}_c \delta^c{}_d$

Solution: The free index has been mislabelled. Possible correction: $\delta^a{}_b = \delta^a{}_c \delta^c{}_b$.

v.) $\phi = (x^a A_a)(y^a B_a)$

Solution: There is ambiguity to the summation, as the indices that are summed over area repeated. Possible correction: $\phi = (x^a A_a)(y^b B_b)$.

2. We define a (contravariant) vector v^a as a quantity that transforms as

$$v'^a = \Lambda^a{}_b v^b, \quad \Lambda^a{}_b \equiv \frac{\partial x'^a}{\partial x^b},$$

under a coordinate transformation $\{x\} \mapsto \{x'\}$. By demanding that the scalar $\phi = v^a \omega_a$ is invariant under coordinate transformations, derive the transformation properties of one-forms ω_a. These are often referred to as *covariant vectors*.

Solution: Requiring that scalar quantities are invariant under coordinate transformations yields:

$$\phi = v^c \omega_c = v'^a {\omega'}_a = \Lambda^a{}_b v^b {\omega'}_a .$$

Then, multiplying throughout by $(\Lambda^{-1})^c_a$,

$$(\Lambda^{-1})^c_a v^c \omega_c = (\Lambda^{-1})^c_a \Lambda^a{}_b v^b {\omega'}_a = \delta^c{}_b v^b {\omega'}_a = v^c {\omega'}_a .$$

This means that

$${\omega'}_a = (\Lambda^{-1})^c_a \omega_c = \frac{\partial x^c}{\partial x'^a} \omega_c ,$$

i.e. that one-forms (or covariant vectors) have the 'inverse' transformation law to contravariant vectors.

3. The action of the covariant derivative on a vector v^a can be written as

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c ,$$

where $\Gamma^a{}_{bc}$ are known as the *Christoffel symbols* or *connection coefficients*.

i.) What is the action of the covariant derivative ∇_a on a scalar? Use this to show how ∇_a must act on a one-form ω_a.

Solution: For a scalar ϕ, we have that $\nabla_a \phi = \partial_a \phi$. Thus:

$$\begin{aligned}
\partial_a (v^b \omega_b) &= \nabla_a (v^b \omega_b) \\
v^b (\partial_a \omega_b) + \omega_b (\partial_a v^b) &= v^b (\nabla_a \omega_b) + \omega_b (\partial_a v^b + \Gamma^b{}_{ac} v^c) \\
v^b (\partial_a \omega_b) &= v^b (\nabla_a \omega_b) + \Gamma^b{}_{ac} \omega_b v^c ,
\end{aligned}$$

Re-labelling indices in the last term, $\Gamma^b{}_{ac} \omega_b v^c = \Gamma^c{}_{ab} \omega_c v^b$, it follows that

$$\nabla_a \omega_b = \partial_a \omega_b - \Gamma^c{}_{ab} \omega_c .$$

ii.) By demanding that the covariant derivative transforms as a $(1,1)$ tensor, show that the Christoffel symbols must transform as

$${\Gamma'}^b{}_{ac} = \frac{\partial x^e}{\partial x'^c} \frac{\partial x^c}{\partial x'^a} \frac{\partial x'^b}{\partial x^d} \Gamma^d{}_{ce} - \frac{\partial x^e}{\partial x'^c} \frac{\partial x^c}{\partial x'^a} \frac{\partial x'^b}{\partial x^c \partial x^e} .$$

Solution: If we demand that $\nabla_a v^b$ transforms as a tensor, we have that

$$\begin{aligned}\nabla'_a v'^b &= \frac{\partial x^c}{\partial x'^a}\frac{\partial x'^b}{\partial x^d}\nabla_c v^d \\ \partial'_a v'^b + \Gamma'^b{}_{ac}v'^c &= \frac{\partial x^c}{\partial x'^a}\frac{\partial x'^b}{\partial x^d}\left(\partial_c v^d + \Gamma^d{}_{ce}v^e\right) \\ \frac{\partial x^c}{\partial x'^a}\frac{\partial}{\partial x^c}\left(\frac{\partial x'^b}{\partial x^d}v^d\right) + \Gamma'^b{}_{ac}\frac{\partial x'^c}{\partial x^e}v^e &= \frac{\partial x^c}{\partial x'^a}\frac{\partial x'^b}{\partial x^d}\frac{\partial v^d}{\partial x^c} + \frac{\partial x^c}{\partial x'^a}\frac{\partial x'^b}{\partial x^d}\Gamma^d{}_{ce}v^e \\ \frac{\partial x^c}{\partial x'^a}\frac{\partial^2 x'^b}{\partial x^c \partial x^e}v^e + \Gamma'^b{}_{ac}\frac{\partial x'^c}{\partial x^e}v^e &= \frac{\partial x^c}{\partial x'^a}\frac{\partial x'^b}{\partial x^d}\Gamma^d{}_{ce}v^e\end{aligned}$$

Multiplying throughout by $\partial x^e/\partial x'^c$ and rearranging, the desired result follows.

iii.) The *Levi-civita connection* is said to be metric compatible ($\nabla_a g_{bc} = 0$) and torsion free ($\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$). By considering cyclic permutations, show that these constraints imply

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right).$$

Solution: Using the condition of metric compatibility,

$$\begin{aligned}\nabla_a g_{bc} &= \partial_a g_{bc} - \Gamma^d{}_{ab}g_{dc} - \Gamma^d{}_{ac}g_{bd} \\ &= \partial_b g_{ca} - \Gamma^d{}_{bc}g_{da} - \Gamma^d{}_{ba}g_{cd} \\ &= \partial_c g_{ab} - \Gamma^d{}_{ca}g_{bd} - \Gamma^d{}_{cb}g_{ad}.\end{aligned}$$

Taking the sum of the second and third lines minus the first, we have that

$$0 = -\partial_a g_{bc} + \partial_b g_{ca} + \partial_c g_{ab} - 2\Gamma^d{}_{bc}g_{ad},$$

where we have also made free of the torsion free condition. Re-arranging the above expression, contracting with g^{ae} and relabelling indices, the desired result follows. In case this was not obvious, the metric is always symmetric in its lower indices.

4. Consider a static spacetime with metric

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = g_{00}\mathrm{d}t^2 + 2g_{0i}\mathrm{d}t\mathrm{d}x^i + g_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where the indices i and j refer to spatial components of the metric. Using the normalisation condition $u^a u_a = -1$, find the four-velocity u^a of a static observer in this spacetime.

Consider a spatial hypersurface such that $u_a \mathrm{d}x^a = 0$. Using this, show that the interval on such a hypersurface is given by

$$\mathrm{d}s^2_{(n-1)} = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \text{where} \quad \gamma_{ij} = g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}.$$

Furthermore, by considering $g_{ab}g^{bc} = \delta_a{}^c$, or otherwise, show that $\gamma^{ij} = g^{ik}g^{j\ell}\gamma_{k\ell} = g^{ij}$. Interpret the meaning of these results with reference to the static observer u^a.

Solution: For a static observer,

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau} = \left(\dot{t}, 0\right), \quad \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

Imposing the normalisation condition

$$-1 = u^a u_a = g_{ab}u^a u^b = g_{00}(u^0)^2 = g_{00}\dot{t}^2 \quad \Rightarrow \quad u^a = \frac{1}{\sqrt{-g_{00}}}(1, 0).$$

Now, the interval can be written as

$$\begin{aligned}\mathrm{d}s^2 &= g_{00}\mathrm{d}t^2 + 2g_{0i}\mathrm{d}t\mathrm{d}x^i + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ &= g_{00}\left(\mathrm{d}t + \frac{g_{0i}}{g_{00}}\mathrm{d}x^i\right)^2 + \left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)\mathrm{d}x^i\mathrm{d}x^j.\end{aligned}$$

Now, the spatial hypersurface is defined by

$$0 = u_a\mathrm{d}x^a = g_{ab}u^b\mathrm{d}x^a = u^0(g_{00}\mathrm{d}t + g_{0i}\mathrm{d}x^i).$$

Using this in the above expression for the interval, the first term vanishes, and the required result follows.

For spatial indices i, j, k and ℓ, we have that

$$g_{kb}g^{bi} = g_{kj}g^{ij} + g_{k0}g^{i0} = \delta_k{}^i, \quad g_{0b}g^{bi} = g_{0j}g^{ij} + g_{00}g^{i0} = 0 \quad \Rightarrow \quad g^{i0} = -\frac{g_{0j}}{g_{00}}g^{ij}.$$

Substituting the last of these expressions into the first,

$$g^{ij}\left(g_{aj} - \frac{g_{0j}g_{0k}}{g_{00}}\right) = g^{ij}\gamma_{jk} = \delta_k{}^i,$$

implying that

$$\gamma^{ij} = g^{ik}g^{j\ell}\gamma_{k\ell} = g^{ik}\delta_k{}^j = g^{ij},$$

as required. γ_{ij} is the induced spatial metric experienced by static observers within the spacetime defined by g_{ab}, with $\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$ being the invariant interval on this hypersurface. Defined in this way, γ^{ij} is the inverse of γ_{ij}.

5. Consider the metric for 3-dimensional Minkowski space (t', r', ϕ'):

$$\mathrm{d}s^2 = -\mathrm{d}t'^2 + \mathrm{d}r'^2 + r'^2\mathrm{d}\phi'^2.$$

i.) We perform a coordinate transformation to a frame which is rotating with a constant angular velocity Ω:

$$t' = t, \quad r' = r, \quad \phi' = \phi + \omega t.$$

What is the metric in the rotating frame (t, r, ϕ)?

Solution: From the coordinate transformation given, $\mathrm{d}\phi' = \mathrm{d}\phi + \omega \mathrm{d}t$, so the metric in the rotating frame is

$$\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t'^2 + \mathrm{d}r'^2 + r'^2\mathrm{d}\phi'^2 \\
&= -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\phi^2 + \omega^2\mathrm{d}t^2 + 2\omega\mathrm{d}\phi\mathrm{d}t) \\
&= -(1 - r^2\omega^2)\mathrm{d}t^2 + \mathrm{d}r^2 + 2r^2\omega\mathrm{d}\phi\mathrm{d}t + r^2\mathrm{d}\phi^2.
\end{aligned}$$

ii.) Consider a static observer in the rotating frame who is located at the position $(r, \phi) = (R, 0)$. How is the proper time of this observer related to the coordinate time t? Explain the physical significance of this result.

Solution: Consider a static observer with proper time τ, so $\mathrm{d}r = \mathrm{d}\phi = 0$. Then,

$$-\mathrm{d}\tau^2 = -(1 - R^2\omega^2)\mathrm{d}t^2 \quad \Rightarrow \quad \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1 - R^2\omega^2}.$$

We see that the proper time of the static observer is related to the coordinate time (in the non-rotating frame) by a time-dilation factor $\sqrt{1 - v^2}$, where $v =$ speed of circular orbit $= R\omega$.

iii.) Compute the four-acceleration $A^a = u^b \nabla_b u^a$ for this static observer with four-velocity u^a. Explain the physical significance of this result.

Solution: The four velocity of this stationary

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau} = (\dot{t}, 0) = \frac{\mathrm{d}t}{\mathrm{d}\tau}(1,0) = \frac{1}{\sqrt{1-R^2\omega^2}}(1,0).$$

Then, using the definition of the four-acceleration,

$$A^a = u^b \nabla_b u^a = u^b(\partial_b u^a + \Gamma^a{}_{bc}u^c) = u^0(\partial_0 u^a + \Gamma^a{}_{00}u^0) = \Gamma^a{}_{00}(u^0)^2,$$

since the time derivative of u^a vanishes. Then, by symmetry, the only non-zero Christoffel symbol occurs for $a = r$, vis:

$$\Gamma^r{}_{00} = \frac{1}{2}g^{rd}(\partial_0 g_{d0} + \partial_0 g_{0d} - \partial_d g_{00}) = -\frac{1}{2}g^{rd}\partial_d g_{00} = -\frac{1}{2}g^{rr}\partial_r g_{00} = -r\omega^2.$$

This gives a four-acceleration of

$$A^a = -\frac{R\omega^2}{1-R^2\omega^2}(0,1,0,0).$$

This result makes physical sense. For $R\omega \ll 1$, this simply reduces to the Newtonian result $A^a = -R\omega^2(0,1,0,0)$. There is a divergence as $R\omega \to 1$ as the (massive) observer would have to experience infinite acceleration to move at such a speed in circular orbit.

iv.) Using your results from Question 4, compute the induced spatial metric γ_{ij} for observers rotating with angular velocity ω in 3-dimensional Minkowski space. Using this, compute the circumference of a circle with radius R as measured by these observers, and explain the physical significance of the result.

Solution: Considering the components of γ_{ij}:

$$\gamma_{rr} = g_{rr} = 1, \quad \gamma_{\phi\phi} = g_{\phi\phi} - \frac{g_{0\phi}^2}{g_{00}} = \frac{r^2}{1-r^2\omega^2}, \quad \gamma_{r\phi} = 0.$$

The circumference of a circle measured by these observers would be

$$\ell = \int \mathrm{d}s = \int_0^{2\pi} \mathrm{d}\phi\, \sqrt{\gamma_{\phi\phi}} = \frac{R}{\sqrt{1-R^2\omega^2}}\int_0^{2\pi}\mathrm{d}\phi = \frac{2\pi R}{\sqrt{1-R^2\omega^2}}.$$

This represents length contraction to the rotational motion; for $R\omega \ll 1$, this reduces to the normal $\ell = 2\pi R$.

6. Let us start from a global inertial frame in Minkowski space (t, x, y, z). Now consider the transformation to a *non-inertial* frame (t', x', y', z') such that

$$t = \left(\frac{1}{g} + z'\right) \sinh(gt'), \quad x = x', \quad y = y', \quad z = \left(\frac{1}{g} + z'\right) \cosh(gt') - \frac{1}{g},$$

for some constant g.

i.) For $gt' \ll 1$, show that this transformation corresponds to a uniformly accelerated reference frame in Newtonian mechanics.

Solution: We expand the frame transformations for $gt' \ll 1$. The x and y coordinates remain unchanged, while

$$t = \left(\frac{1}{g} + z'\right) \left(gt' + \frac{1}{3}(gt')^3 + \ldots\right) \simeq t',$$
$$z = \left(\frac{1}{g} + z'\right) \left(1 + \frac{1}{2}(gt')^2 + \ldots\right) - \frac{1}{g} \simeq z' + \frac{1}{2}gt'^2.$$

Thus, we obtain the familiar Galilean transformation

$$t = t', \quad x = x', \quad y = y', \quad z = z' + \frac{1}{2}gt'^2,$$

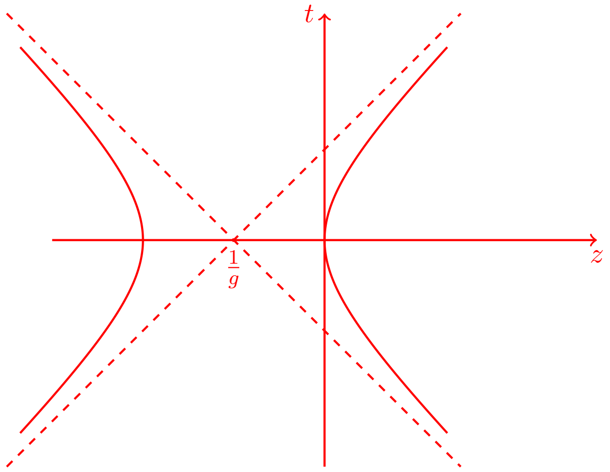
i.e. all of the coordinates are unchanged except via a shift along the relative direction of motion of the two frames.

ii.) Plot the trajectory of the point $z' = 0$ in the inertial frame.

Solution: Without loss of generality, we set $x = y = 0$. Then,

$$t = \frac{1}{g}\sinh(gt'), \quad z = \frac{1}{g}\cosh(gt') - \frac{1}{g} \quad \Rightarrow \quad (1 + gz)^2 - (gt)^2 = 1.$$

This means that the worldline of the relevant trajectory in the inertial frame is hyperbolic. The plot should feature intercepts at $z = 0$ and $z = -2/g$, and asymptotes along $t = \pm(z + 1/g)$.



iii.) Show that a clock at rest at $z' = h$ runs fast compared to a clock at rest at $z' = 0$ by the factor $(1 + gh)$, as observed in the inertial frame. Use the equivalence principle to interpret this result in terms of gravitational time dilation.

Solution: Let the coordinate times t_1 and t_2 correspond to $z' = 0$ and $z' = h$ respectively. Then, for a time interval $\Delta t'$ in the non-inertial frame,

$$\frac{\Delta t_2}{\Delta t_1} \simeq \frac{\left(\frac{1}{g} + h\right)\sinh(g\Delta t')}{\frac{1}{g}\sinh(g\Delta t')} = 1 + gh.$$

Thus, we find that

$$\Delta t_2(z' = h) = (1 + gh)\Delta t_1(z' = 0),$$

meaning that a clock at rest at $z' = h$ runs fast compared to a clock at rest at $z' = 0$ by a factor of $(1 + gh)$ (in the inertial frame).

Now, the equivalence principle states that in the vicinity of some point on a manifold under the presence of gravity that a falling observer may set up some local inertial frame in which the metric is locally Minkowski. This is essentially a statement that gravitational fields are indistinguishable from acceleration. Given that we are considering a uniformly accelerated, non-inertial frame, we can interpret the results in terms of gravitational time dilation by writing that

$$\text{Rate Received} = (1 - (\Phi_2 - \Phi_1)) \times \text{Rate Emitted},$$

where Φ_1 and Φ_2 are the local Newtonian gravitational potentials at two given points. In this case, $\Phi = (mgh)/m = gh$, which gives the same result as above.

iv.) What is the line element $\mathrm{d}s^2$ of a uniform gravitational field?

Solution: Using the total derivatives

$$\mathrm{d}t = \sinh(gt')\mathrm{d}z' + (1 + gz')\cosh(gt')\mathrm{d}t',$$
$$\mathrm{d}z = \cosh(gt')\mathrm{d}z' + (1 + gz')\sinh(gt')\mathrm{d}t',$$

we note that

$$-\mathrm{d}t^2 + \mathrm{d}z^2 = \left[-(1 + gz')^2\mathrm{d}t'^2 + \mathrm{d}z'^2\right]\left[\cosh^2(gt') - \sinh^2(gt')\right] = -(1 + gz')^2\mathrm{d}t'^2 + \mathrm{d}z'^2.$$

Using these results in the Minkowski metric, we thus have that

$$\mathrm{d}s^2 = -(1 + gz')^2\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2,$$

which confirms our previous statement that $\Phi = gh$.

7. The energy-momentum tensor of a perfect fluid in Minkowski space is given by

$$T^{ab} = p\eta^{ab} + (p+\rho)u^a u^b,$$

where u^a is the four-velocity of the fluid, and $\eta^{ab} = \mathrm{diag}(-1,1,1,1)$. By considering an observer at rest with respect to the motion of the fluid, explain the physical meaning of p and ρ.

Solution: In the rest frame of the fluid $u^a = u^b = (1,0)$. This means that $T^{00} = \rho$ and $T^{ii} = p$; i.e. ρ is the local mass (energy) density, while p is the pressure of the fluid.

The equation of motion of a perfect fluid in a local inertial frame is

$$\partial_a T^{ab} = 0. \tag{1}$$

The remainder of this question is devoted to deriving the equations of fluid mechanics from this one expression. To begin, show that the tensor

$$h^a{}_b = \delta^a{}_b + u^a u_b,$$

satisfies $h^a{}_b u^b = 0$, $h^a{}_b h^b{}_c = h^a{}_c$ and $h^a{}_a = 3$, and therefore explain why $h^a{}_b$ is a projector onto the three-dimensional hypersurfaces perpendicular to u^a. What is the meaning of the tensor $h_{ab} = \eta_{ac}h^c{}_b$? By projecting (1) parallel and perpendicular to the four-velocity u^a, show that

$$\partial_a(\rho u^a) + p\partial_a u^a = 0, \quad (p+\rho)(u^b\partial_b)u^a + h^{ab}\partial_b p = 0. \tag{2}$$

In the Newtonian limit, we approximate that $u^i \ll u^0$, $p \ll \rho$ and $|\mathbf{u}|(\partial p/\partial t) \ll |\nabla p|$. What is the physical intuition behind each of these approximations? Using these, show that (2) reduces to the familiar fluid equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \rho\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} = -\nabla p.$$

Solution: Consider the following identities on $h^a{}_b$:

$$h^a{}_b u^b = u^a(1 + u_b u^b) = 0,$$
$$h^a{}_b h^b{}_c = \delta^a{}_b\delta^b{}_c + \delta^a{}_b u^b u_c + \delta^b{}_c u^a u_b + u^a u_b u^b u_c = \delta^a{}_c + u^a u_c = h^a{}_c,$$
$$h^a{}_a = \delta^a{}_a + u^a u_a = 3.$$

The tensor $h^a{}_b$ is thus clearly a projector onto the three-dimensional hypersurfaces perpendicular to u^a, as it is perpendicular to u^a, multiple applications of the projector yields the identity (once you are in the projected space, the projector does nothing), and the value of its trace is 3. The tensor h_{ab} is then the induced metric on the orthogonal hypersurface.

A useful identity for these calculations will be that

$$0 = \partial_a(u_b u^b) = 2u_b\partial_a u^b.$$

For the parallel projection:

$$\begin{aligned}
0 &= u_b\partial_a T^{ab} \\
&= u^a\partial_a p + u_b u^a u^b \partial_a(p+\rho) + (p+\rho)u_b\partial_a(u^a u^b) \\
&= u^a\partial_a p - u^a\partial_a(p+\rho) + (p+\rho)u_b u^b\partial_a u^a \\
&= -u^a\partial_a\rho - \rho\partial_a u^a - p\partial_a u^a,
\end{aligned}$$

which is the desired result. For the perpendicular projection,

$$\begin{aligned}
0 &= h^c{}_b\partial_a T^{ab} \\
&= h^c{}_b\left[\eta^{ab}\partial_a p + u^a u^b\partial_a(p+\rho) + (p+\rho)\partial_a(u^a u^b)\right] \\
&= h^{ac}\partial_a p + (p+\rho)h^c{}_b u^a\partial_a u^b \\
&= h^{ac}\partial_a p + (p+\rho)(\delta^c{}_b u^a\partial_a u^b + u^c u_b u^a\partial_a u^b) \\
&= h^{ac}\partial_a p + (p+\rho)u^a\partial_a u^c.
\end{aligned}$$

A relabelling of indices yields the desired result.

In the Newtonian limit, we have the following approximations:

- $u^i \ll u^0$: Non-relativistic motion, speeds much less than the speed of light.
- $p \ll \rho$: Newtonian fluids are pressureless in their own rest frame.
- $|\mathbf{u}|(\partial\rho/\partial t) \ll |\nabla p|$: Static pressure gradients/sub-sonic limit. Pressure gradients to not change quickly in comparison to their magnitude.

For the continuity equation, we have that

$$\partial_0(\rho u^0) + \partial_i(\rho u^i) + p\partial_i u^i \simeq \partial_0(\rho u^0) + \partial_i(\rho u^i) = u^0\partial_0\rho + \partial_i(\rho u^i),$$

which is the desired result. For the Euler equation, we consider the spatial $a = i$ component:

$$(p+\rho)(u^b\partial_b)u^i + h^{ib}\partial_b p \simeq \rho(u^b\partial_b)u^i + h^{ib}\partial_b p,$$

since $p \ll \rho$. Now, $u^b\partial_b u^i = \partial_0 u^i + u^j\partial_j u^i$, and $h^{ib}\partial_b p = \partial^i p$, and the result follows. However, we have assumed properties about the induced metric h_{ab}. Alternatively, we may write

$$h^{ib}\partial_b p = h^a{}_b\partial^b p = h^i{}_0\partial^0 p + h^i{}_j\partial^j p \simeq h^i{}_j\partial^j p = \delta^i{}_j\partial^j p + u^i u_j\partial^j p \simeq \partial^i p,$$

and the desired result also follows.

Problem Set 2: Geodesics, Curvature and Schwarzschild

8. Consider a curve $x^a(\lambda)$ in a metric space g_{ab} parametrised by some (real) affine parameter λ. What is the condition for this curve to be time-like? By considering variations of the functional

$$S = \int \mathrm{d}\tau = \int \mathrm{d}\lambda \, \mathcal{L}(x, \dot{x}, \lambda), \quad \mathcal{L} = \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \sqrt{-g_{ab}\dot{x}^a\dot{x}^b}, \tag{3}$$

show that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{ac}\dot{x}^a\right) = \frac{1}{2}(\partial_c g_{ab})\dot{x}^a\dot{x}^b, \tag{4}$$

where we recall that the dot indicates differentiation with respect to the affine parameter λ. This is the *geodesic equation*.

Solution: The condition for a timelike curve is that $g_{ab}\dot{x}^a\dot{x}^b < 0$. The variation in the Lagrangian is given by

$$\delta\mathcal{L} = \mathcal{L}(x + \delta x, \dot{x} + \delta\dot{x}, \lambda) - \mathcal{L}(x, \dot{x}, \lambda) = \frac{\partial \mathcal{L}}{\partial x^c}\delta x^c + \frac{\partial \mathcal{L}}{\partial \dot{x}^c}\delta\dot{x}^c.$$

Then, the variation in the action is

$$\delta S = \int \mathrm{d}\lambda \left[\frac{\partial \mathcal{L}}{\partial x^c}\delta x^c + \frac{\partial \mathcal{L}}{\partial \dot{x}^c}\delta\dot{x}^c\right] = \int \mathrm{d}\lambda \left[\frac{\partial \mathcal{L}}{\partial x^c} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^c}\right)\right]\delta\dot{x}^c,$$

where we have assumed that any surface terms vanish. The path taken by along the geodesic curve will thus satisfy the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^c}\right) = \frac{\partial \mathcal{L}}{\partial x^c}.$$

Now, extremising $\mathcal{L}$ is equivalent to extremising $\mathcal{L}^2$, and so we instead consider $\mathcal{L} = -g_{ab}\dot{x}^a\dot{x}^b$:

$$\frac{\partial \mathcal{L}}{\partial x^c} = -(\partial_c g_{ab})\dot{x}^a\dot{x}^b, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}^c} = -2g_{ac}\dot{x}^a,$$

from which the result follows.

A geodesic curve is defined as one that parallel transports its own tangent vector. From this, show that an alternative expression for the geodesic equation is

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0. \tag{5}$$

By direct calculation, show that (4) is equivalent to (5). Lastly, by considering the total derivative of the Hamiltonian

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{x}^a}\dot{x}^a - \mathcal{L},$$

show that the Hamiltonian is conserved for the Lagrangian defined in (3). What is the condition for it to be conserved for a general Lagrangian $\mathcal{L}$?

Solution: Starting from the condition for parallel transport,

$$0 = \dot{x}^b \nabla_b \dot{x}^a = \dot{x}^b(\partial_b \dot{x}^a + \Gamma^a{}_{bc}\dot{x}^c) = \ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c,$$

as required. Starting from (4),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{ac}\dot{x}^a) - \frac{1}{2}(\partial_c g_{ab})\dot{x}^a\dot{x}^b &= g_{ac}\ddot{x}^a + (\partial_b g_{ac})\dot{x}^a\dot{x}^b - \frac{1}{2}\partial_c g_{ab}\dot{x}^a\dot{x}^b \\ &= g_{ac}\ddot{x}^a + \frac{1}{2}(\partial_b g_{ac} + \partial_a g_{bc} - \partial_c g_{ab})\dot{x}^a\dot{x}^b.\end{aligned}$$

Relabelling a to d in the last expression,

$$\begin{aligned}&\frac{1}{2}(\partial_b g_{ac} + \partial_a g_{bc} - \partial_c g_{ab})\dot{x}^a\dot{x}^b = \frac{1}{2}(\partial_b g_{dc} + \partial_d g_{bc} - \partial_c g_{db})\dot{x}^d\dot{x}^b \\ &= \frac{1}{2}g_{ac}g^{ae}(\partial_b g_{ed} + \partial_d g_{eb} - \partial_e g_{bd})\dot{x}^b\dot{x}^d = g_{ac}\Gamma^a{}_{bd}\dot{x}^b\dot{x}^d.\end{aligned}$$

Thus,

$$g_{ac}\left(\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^b\dot{x}^d\right) = 0 \quad \Rightarrow \quad \ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0,$$

where we have relabelled indices in the last expression. Note that the equivalence of (4) and (5) makes calculation of the Christoffel symbols relatively straightforward, as they can simply be read off as the coefficient of $\dot{x}^a\dot{x}^b$ when evaluating (4).

Consider the total derivative of the Hamiltonian $\mathcal{H}$ with respect to the affine parameter λ:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\dot{x}^a\right) - \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\dot{x}^a\right) - \left[\frac{\partial \mathcal{L}}{\partial \lambda} + \frac{\partial \mathcal{L}}{\partial x^a}\dot{x}^a + \frac{\partial \mathcal{L}}{\partial \dot{x}^a}\ddot{x}^a\right].$$

Noting that

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\ddot{x}^a = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\dot{x}^a\right) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right)\dot{x}^a$$

and that $\mathcal{L}$ satisfies the Euler-Lagrange equations, it follows that

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\lambda} = -\frac{\partial \mathcal{L}}{\partial \lambda}.$$

Thus, the Hamiltonian is conserved if $\mathcal{L}$ does not depend explicitly on λ. This is true for the case of $\mathcal{L} = -g_{ab}\dot{x}^a\dot{x}^b$. It is also the case that $\mathcal{H} = \mathcal{L}$ for this Lagrangian, meaning that $\mathcal{L}$ is also conserved along geodesics. This fact will prove useful when analysing geodesic motion.

9. The Lie derivative of a $(2,0)$ tensor with respect to a vector field x^a is given by

$$\mathcal{L}_x T_{ab} = (x^c\partial_c)T_{ab} + (\partial_a x^c)T_{cb} + (\partial_b x^c)T_{ac}. \tag{6}$$

Show that you can replace any ∂_a with any covariant derivative ∇_a in this expression, and so argue that the Lie derivative transforms as a tensor.

Solution: Replacing ∂_a with ∇_a throughout the definition of the total derivative, we have

$$\begin{aligned}\mathcal{L}_x T_{ab} &= (x^c\nabla_c)T_{ab} + (\nabla_a x^c)T_{cb} + (\nabla_b x^c)T_{ac}\\ &= x^c\left(\partial_c T_{ab} - \Gamma^d{}_{ca}T_{db} - \Gamma^d{}_{cb}T_{ad}\right) + \left(\partial_a x^c + \Gamma^c{}_{ae}x^e\right)T_{cb} + \left(\partial_b x^c + \Gamma^c{}_{be}x^e\right)T_{ac}\\ &= (x^c\partial_c)T_{ab} + (\partial_a x^c)T_{cb} + (\partial_b x^c)T_{ac}\\ &\qquad - x^c\Gamma^d{}_{ca}T_{db} + x^e\Gamma^c{}_{ae}T_{cb} - x^c\Gamma^d{}_{cb}T_{ad} + x^e\Gamma^c{}_{be}T_{ac}\\ &= (x^c\partial_c)T_{ab} + (\partial_a x^c)T_{cb} + (\partial_b x^c)T_{ac},\end{aligned}$$

where we have assumed a torsion free connection, and relabelled contracted indices. The Lie derivative must this transform as a tensor expression.

Consider a vector field K^a that generates a coordinate transformation $x'^a = x^a + \delta x^a = x^a + \epsilon K^a$. Show that

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}g_{cd},$$

remains invariant under this transformation if $\mathcal{L}_x g_{ab} = 0$. Show that this condition is equivalent to $\nabla_a K_b + \nabla_b K_a = 0$ for a metric compatible connection ∇_a.

Solution: Using the transformation, we note that

$$\begin{aligned}\frac{\partial x^c}{\partial x'^a} &= \frac{\partial}{\partial x'^a}(x'^c - \delta x^c) = \delta^c{}_a - \partial_a\delta x^c + \mathcal{O}((\delta x^c)^2)\\ \frac{\partial x^d}{\partial x'^b} &= \frac{\partial}{\partial x'^b}(x'^d - \delta x^d) = \delta^d{}_b - \partial_b\delta x^d + \mathcal{O}((\delta x^d)^2)\\ g'_{ab}(x') &= g_{ab}(x) + \partial_c g_{ab}\,\delta x^c + \mathcal{O}((\delta x^c)^2).\end{aligned}$$

Using these in the transformation condition for the metric:

$$\begin{aligned}g'_{ab} - \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}g_{cd} &= g_{ab} + \delta x^c\partial_c g_{ab} - \left(\delta^c{}_a\delta^d{}_b - \delta^c{}_a\partial_b\delta x^d - \delta^d{}_b\partial_a\delta x^c\right)g_{cd}\\ 0 &= \left[(K^c\partial_c)g_{ab} + (\partial_a K^c)g_{cb} + (\partial_b K^c)g_{ac}\right]\epsilon\\ \Rightarrow \mathcal{L}_K g_{ab} &= 0.\end{aligned}$$

However, we have already shown than we can replace all partial derivatives by covariant derivatives in the definition of the lie derivative. Noting that $K^c\nabla_c g_{ab} = 0$ by the metricity condition, it follows that $\nabla_a K_b + \nabla_b K_a = 0$ by lowering the indices on K^c with the metric.

This condition can also be derived by considering variations of the functional

$$S = \int \mathrm{d}\lambda \, g_{ab}\dot{x}^a \dot{x}^b,$$

vis.:

$$\begin{aligned}
\delta S &= \int \mathrm{d}\lambda \, \Big[(K^c \partial_c g_{ab})\dot{x}^a \dot{x}^b + g_{ab}(\dot{K}^a \dot{x}^b + \dot{x}^a \dot{K}^b)\Big] \\
&= \int \mathrm{d}\lambda \, \Big[(K^c \partial_c g_{ab})\dot{x}^a \dot{x}^b + g_{ab}(\partial_c K^a)\dot{x}^c \dot{x}^b + g_{ab}(\partial_c K^b)\dot{x}^a \dot{x}^c\Big] \\
&= \int \mathrm{d}\lambda \, \mathcal{L}_K g_{ab}\dot{x}^a \dot{x}^b,
\end{aligned}$$

upon a relabelling of indices.

A vector field satisfying this property is known as a 'Killing vector', and generates an infinitesimal symmetry of the geometry defined by the metric g_{ab}. Show that the inner product $K_b \dot{x}^b$ is conserved along geodesics.

Solution: Consider the parallel transport of $K_b \dot{x}^b$ along some $x^c(\lambda)$:

$$\dot{x}^c \nabla_c (K_b \dot{x}^b) = \dot{x}^c \nabla_c (g_{ab} K^a \dot{x}^b) = K_b(\dot{x}^c \nabla_c \dot{x}^b) + \dot{x}^b \dot{x}^c \nabla_c K_b = 0,$$

where the first term is zero by the definition of the geodesic equation, and the second is zero since it is a product of a symmetric and antisymmetric quantity. Thus, we conclude that $K_b \dot{x}^b$ = constant along geodesics. For example, Schwarzschild spacetime has two Killing vectors $K^0 = (1, 0, 0, 0)$ and $K^\phi = (0, 0, 0, 1)$ which correspond to time invariance and rotational symmetry.

10. For (contravariant) vectors, the Riemann curvature tensor is defined as

$$[\nabla_c, \nabla_d]\, v^a = (\nabla_c \nabla_d - \nabla_d \nabla_c)\, v^a = R^a{}_{bcd} v^b. \tag{7}$$

Considering the left-hand side of this expression, show explicitly that

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}.$$

By evaluating $R^a{}_{bcd}$ in local inertial coordinates, or otherwise, show that for a metric compatible and torsion free connection $R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc}$ and $R_{a[bcd]} = R_{abcd} + R_{acdb} + R_{adbc} = 0$. Why is it sufficient to use local inertial coordinates to prove these identities?

Solution: Consider

$$\begin{aligned}\nabla_c\nabla_d v^a &= \nabla_c(\partial_d v^a + \Gamma^a{}_{db}v^b)\\ &= \partial_c\partial_d v^a + \partial_c(\Gamma^a{}_{db}v^b) + \Gamma^a{}_{ce}\partial_d v^e - \Gamma^e{}_{cd}\partial_e v^a + \Gamma^a{}_{ce}\Gamma^e{}_{db}v^b - \Gamma^e{}_{cd}\Gamma^a{}_{eb}v^b\\ &= \partial_c(\Gamma^a{}_{db}v^b) + \Gamma^a{}_{ce}\partial_d v^e + \Gamma^a{}_{ce}\Gamma^e{}_{db}v^b + \text{terms symmetric under } c \leftrightarrow d.\end{aligned}$$

Then:

$$\begin{aligned}&(\nabla_c\nabla_d - \nabla_d\nabla_c)v^a\\ &= \partial_c(\Gamma^a{}_{db}v^a) + \Gamma^a{}_{ce}\partial_d v^e + \Gamma^a{}_{ce}\Gamma^e{}_{db}v^b - \partial_d(\Gamma^a{}_{cb}v^b) - \Gamma^a{}_{de}\partial_c v^e - \Gamma^a{}_{de}\Gamma^e{}_{cb}v^b\\ &= \left[\partial_c\Gamma^a{}_{db} - \partial_d\Gamma^a{}_{cb} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}\right]v^b,\end{aligned}$$

as required.

In local inertial coordinates ξ^a, $g_{ab} \simeq \eta_{ab}$ and $\Gamma = 0$, $\partial\Gamma \neq 0$, meaning that in local inertial coordinates,

$$R^a{}_{bcd}|_\xi = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} = \frac{1}{2}\eta^{ae}(\partial_c\partial_d g_{ed} + \partial_e\partial_d g_{bc} - \partial_e\partial_c g_{bd} - \partial_b\partial_d g_{ec}),$$

so

$$R_{abcd}|_\xi = \frac{1}{2}(\partial_c\partial_b g_{ad} + \partial_a\partial_d g_{bc} - \partial_a\partial_c g_{bd} - \partial_b\partial_d g_{ac}).$$

From this expression it is clear that $R_{abcd}|_\xi = R_{cdab}|_\xi = -R_{bacd}|_\xi = -R_{abdc}|_\xi$ and $R_{a[bcd]}\big|_\xi = 0$. We have thus proven all of the given identities in local inertial coordinates. However, given that all of these expressions are tensorial, they must hold regardless of the choice of frame, meaning that they must be valid in all frames, rather than just local inertial coordinates.

For a metric compatible and torsion free connection, the Riemann tensor also satisfies

$$R_{ab[cd;e]} = R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0.$$

How many independent components does the Riemann tensor have in n dimensions? You should find that it only has one independent component for $n = 2$. Prove that the Riemann tensor must then take the form

$$R_{abcd} = \frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

where R is the Ricci scalar. Hence show that the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ vanishes in two-dimensions.

Solution: R_{abcd} is symmetric under the exchange of ab and cd, giving rise to $m(m+1)/2$ components, where m is the number of independent components in the ab or cd subset. However, R_{abcd} is symmetric under the exchange of the indices $a \leftrightarrow b$, $c \leftrightarrow d$, implying that $m = n(n-1)/2$. However, the first Bianchi identity $R_{a[bcd]} = 0$ provides $\binom{n}{4}$ relationships between these components. This is because, in this identity, all of the indices must be different, otherwise it reduces to the previous symmetry and antistymmetry relations. Thus, we are choosing 4 indices out of a possible n. Thus, the number of independent components N of the Riemann tensor is given by

$$N = \frac{1}{2}m(m+1)\Big|_{m=\frac{1}{2}n(n-1)} - \binom{n}{4} = \frac{1}{12}n^2(n^2-1).$$

It is clear that $N = 1$ for $n = 2$. This means that the Riemann tensor must be completely determined by the Ricci scalar. The only possible form of the Riemann tensor that satisfies all of the required symmetry properties is

$$R_{abcd} = f(R)(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

where $f(R)$ is some function of the Ricci scalar R. Then:

$$R = g^{bd}g^{ac}f(R)(g_{ac}g_{bd} - g_{ad}g_{cb}) = f(R)((g^a{}_a)^2 - g^b{}_b) = 2f(R) \quad \Rightarrow f(R) = \frac{1}{2}R,$$

giving the desired form of the Riemann tensor. Now,

$$R_{ab} = R^c{}_{acb} = g^{dc}R_{dacb} = \frac{1}{2}R(g^{dc}g_{dc}g_{ab} - g^c{}_b g_{ac}) = \frac{1}{2}g_{ab}R \quad \Rightarrow \quad G_{ab} = 0.$$

11. Birkoff's theorem tells us that the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \quad r_s = 2GM \tag{8}$$

is the unique spherically symmetric vacuum solution to the Einstein field equations in the presence of a point mass M. What is the physical meaning of the coordinate time t in this solution? By considering the geodesic equation (4), or otherwise, show that you can always restrict attention to time-like geodesics lying in the equatorial plane, $\theta = \pi/2$.

Solution: The coordinate time t is the proper time measured by an observer at rest, located at $r \to \infty$ relative to the point mass M.

Consider the geodesic equation (4) for $c = \theta$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(2r^2\dot{\theta}) = 2r^2 \sin\theta \cos\theta\, \dot{\phi}^2.$$

Letting $\theta = \pi/2$, it is clear that $\dot{\theta} = \text{constant}/r^2$. We can always choose our coordinates such that $\dot{\theta} = \ddot{\theta} = 0$, meaning that $\theta = \pi/2$ throughout the trajectory.

Show that geodesics in the Schwarschild metric have two conserved quantities

$$E = \left(1 - \frac{r_s}{r}\right)\dot{t} \quad \text{and} \quad J = r^2\dot{\phi}.$$

What symmetries do these correspond to? Then, considering a change of variables $u = 1/r$, show that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + V_{\text{eff}}(u) = \frac{1}{J^2}(E^2 - k^2), \tag{9}$$

where $k = \mathrm{d}\tau/\mathrm{d}\lambda$ and we have defined some *effective potential*

$$V_{\text{eff}}(u) = u^2(1 - r_s u) - \frac{k^2}{J^2} r_s u. \tag{10}$$

Give physical interpretations of each of the terms in (10). For both timelike and null geodesics, sketch V_{eff} as a function of radius r, finding any turning points, and give a description of the type of orbit at different values of r. Finally, find expressions for E and J in the case of a circular orbit, as well as the orbital frequency $\omega(r)$.

Solution: To show that Schwarzschild permits the two conserved quantities E and J, one can notice that (8) permits two Killing vectors ($K^0 = (1,0,0,0)$ and $K^\phi = (0,0,0,1)$) and use the fact that $g_{ab}K^a x^b$ is conserved along geodesics. Alternatively, one can consider the geodesic equation (4); for $c = 0$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{00}\dot{x}^0) = 0 \quad \Rightarrow \quad \left(1 - \frac{r_s}{r}\right)\dot{t} = E,$$

while for $c = \phi$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{\phi\phi}\dot{x}^\phi) = 0 \quad \Rightarrow \quad r^2 \sin^2\theta\, \dot{\phi} = J,$$

with the result following for $\theta = \pi/2$. These correspond to time invariance (implying energy conservation by Noether's theorem) and rotational symmetry (implying the conservation of angular momentum).

It is clear that the interval is conserved throughout the motion. Defining k as in the question, we have that

$$\mathcal{L} = -k^2 = -\left(1 - \frac{r_s}{r}\right)\dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2.$$

Once again moving to $\theta = \pi/2$ without loss of generality, we let $\dot{t} = (1 - r_s/r)^{-1}E$ and $\dot{\phi} = J/r^2$, and use

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\lambda} = \frac{J^2}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi} = -J\frac{\mathrm{d}u}{\mathrm{d}\phi},$$

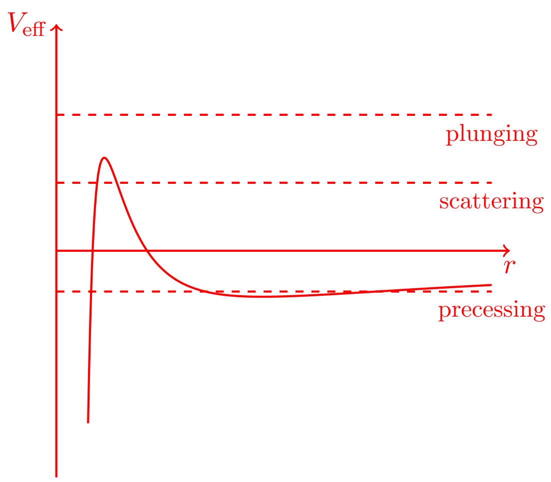
to derive (9). Physically, each of the terms in the effective potential represent

$$J^2 V_{\text{eff}}(r) = \underbrace{\frac{J^2}{r^2}}_{\text{angular momentum barrier}} - \underbrace{\frac{r_s J^2}{r^3}}_{\text{GR correction}} - \underbrace{\frac{r_s k^2}{r}}_{\text{Newtonian gravitational potential}}$$

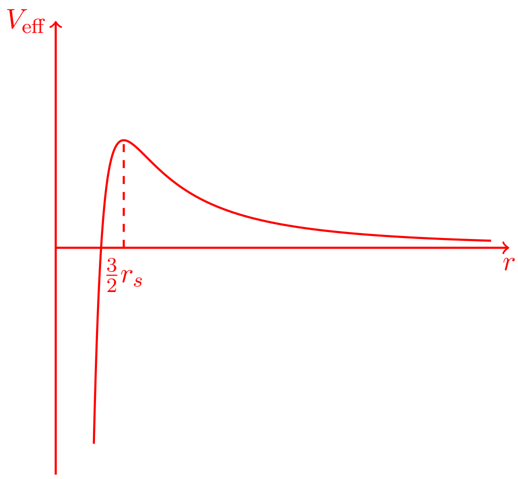
Finding the turning points of V_{eff}, we have that:

$$\frac{\partial V_{\text{eff}}}{\partial u} = 0 \quad \Rightarrow \quad u = \frac{1}{r} = \frac{1}{3r_s}\left[1 \pm \sqrt{1 - 3\left(\frac{kr_s}{J}\right)^2}\right].$$

For timelike geodesics, $k = 1$, the two solutions for u correspond to the extrema of elliptical orbits. For $J = \sqrt{3}r_s$, the two extrema combine, and we have a minimum stable circular orbit for $r = 3r_s$. Below this value of J, all orbits will be bound/plunging orbits.



For null geodesics, $k = 0$, the extrema are $r = 0$ or $r = 3r_s/2$. This means that circular orbits can exist for null geodesics, but they only occur at an unstable maximum. There are no bound orbits.



Finally, we consider timelike circular orbits, for which $k = 1$, $\mathrm{d}r = \mathrm{d}\theta = 0$. From the metric, we have that

$$-1 = -\left(1 - \frac{r_s}{r}\right)\dot{t}^2 + r^2\dot{\phi}^2.$$

Differentiating with respect to r:

$$0 = -\frac{r_s}{r^2}\dot{t}^2 + 2r\dot{\phi}^2 \quad \Rightarrow \quad \omega(r) = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \sqrt{\frac{r_s}{2r^3}} = \sqrt{\frac{GM}{r^3}}.$$

Using the definition of J,

$$J = r^2\dot{\phi} = r^2\dot{t}\omega \quad \Rightarrow \quad J = \left(1 - \frac{r_s}{r}\right)^{-1} r^2\omega E.$$

Substituting this into the orbit equation (9) for $\mathrm{d}u/\mathrm{d}\phi = 0$, we find that

$$E = \frac{(1 - r_s/r)}{\sqrt{1 - 3r_s/2r}}, \quad J = \frac{\sqrt{GMr}}{\sqrt{1 - 3r_s/2r}}.$$

We see that these diverge at $r = 3r_s/2$, corresponding to the fact that there are no stable circular orbits exist below this value.

12. We define the *impact parameter* by

$$b = \frac{J}{\sqrt{E^2 - k^2}}.$$

We wish to consider when incoming geodesics will be captured by the black-hole.

i.) Show that a massless particle is captured by the black-hole if the impact parameter is smaller than a certain critical value $b < b_c$, and find an expression for the capture cross-section $\sigma = \pi b_c^2$ in terms of M.

Solution: From (9) with $k=0$, the distance of closest approach will occur at some value of r such that $\dot{r}=0$, so

$$\frac{E^2}{J^2} = \frac{1}{r^2}\left(1-\frac{r_s}{r}\right) \quad \Rightarrow \quad b^2 = \frac{r^3}{r-r_s}.$$

Now, for the photon to be captured by the black hole, r in this expression must correspond to the minimum circular orbit for a photon. We found this in the previous question to be $r_{\text{min}} = 3r_s/2$, meaning that

$$b_c^2 = \left.\frac{r^3}{r-r_s}\right|_{r=r_{\text{min}}} = \frac{27}{4}r_s^2 \quad \Rightarrow \quad \sigma = \frac{27}{4}\pi r_s^2 = 27\pi G^2 M^2.$$

ii.) Consider a massive particle that starts at $r\to\infty$ with a non-relativistic velocity $v \ll 1$ as measured by a stationary observer. Explain why $b = J/v + \mathcal{O}(v)$, and explain the physical significance of the impact parameter in this case (it may be helpful to use a diagram). Find an expression for b_c in the case of the massive particle, and show that

$$\sigma = \frac{16\pi G^2 M^2}{v^2}.$$

Solution: We assume that the particle is initially stationary at $\to\infty$ apart from vanishingly small velocity $v \ll 1$. As the particle is initially in motion relative to the centre of the system, it must have some non-zero angular momentum $J=\gamma v b$, where v is the usual Lorentz gamma factor. Then we have that

$$b = \frac{J}{\gamma v} = \frac{J}{v}(1-v^2)^{1/2} \simeq \frac{J}{v} + \mathcal{O}(v),$$

as required. b is thus the distance between the parallel lines of its own initial/final trajectory, and that passing through the scattering centre (offset).

As the particle is initially at rest at $r \to \infty$, $E = 1$ since the proper time on the worldline of the particle and that of the stationary (coordinate) observer are the same $(\mathrm{d}r/\mathrm{d}\tau = 1)$. Given that $k = 1$ for a massive particle, we can rewrite our orbit equation as

$$0 = \dot{r}^2 + \frac{J^2}{r^2} - \frac{r_s}{r} - \frac{J^2 r_s}{r^3} \quad \Rightarrow \quad \dot{r}^2 = \left(\frac{r_s}{r}\right)^3 \left(\frac{r^2}{r_s^2} - \frac{J^2}{r_s^2}\frac{r}{r_s} + \frac{J^2}{r_s^2}\right).$$

This can be factorised as

$$\dot{r}^2 = \left(\frac{r_s}{r}\right)^3 \left(\frac{r}{r_s} - \frac{r_+}{r_s}\right)\left(\frac{r}{r_s} - \frac{r_-}{r_s}\right), \quad r_\pm = \frac{J^2}{2r_s^2}\left[1 \pm \sqrt{1 - \frac{4r_s^2}{J^2}}\right].$$

It is clear from this that there are turning points with $\dot{r} = 0$, which only exist for

$$1 - \frac{4r_s^2}{J^2} \geq 0 \quad \Rightarrow \quad J \geq 2r_s.$$

This is the critical value of the angular momentum at which the object will be captured. This means that $b_c = 2r_s/v$, and the result for the capture cross-section follows.

13. Starting from (9), show that

$$u'' + u = \frac{3}{2}r_s u^2 + \frac{r_s}{2J^2}k^2, \quad u' = \frac{\mathrm{d}u}{\mathrm{d}\phi}.$$

i.) By considering perturbations $u = u_0 + \delta u$, $\delta u \ll u_0$ around circular orbits, find expressions for $\delta u(\phi)$ for timelike and null orbits. Using this, show that timelike circular orbits may only exist for $r > (3/2)r_s$, and that they are unstable for $r < 3r_s$. Similarly, show that null orbits are always unstable.

Solution: Denoting $u' = \mathrm{d}u/\mathrm{d}\phi$, we differentiate (9) with respect to ϕ, and divide throughout by u', such that:

$$u'' + u = \frac{3}{2} r_s u^2 + \frac{r_s}{2J^2} k^2.$$

We first consider timelike orbits ($k = 1$). If u_0 corresponds to a circular orbit, $u_0'' = 0$, and so

$$u_0 = \frac{3}{2} r_s u_0^2 + \frac{r_s}{2J^2}.$$

Now, the innermost circular orbit will occur for $J \to \infty$, corresponding to $u_0^{\min} = 2/(3r_s) \Leftrightarrow r_0 = 3r_s/2$. This was the reason for the divergences in the expressions for E and J found in the last part of question 11. Then, letting $u = u_0 + \delta u$, we have that

$$\delta u'' + \delta u = \frac{3}{2} r_s u_0 (2\delta u) = 3 r_s u_0 \delta u.$$

This has solution:

$$\delta u(\phi) = c_1 \cos(\alpha\phi) + c_2 \sin(\alpha\phi), \quad \alpha = \sqrt{1 - 3r_s u_0},$$

for constants c_1 and c_2. Clearly, orbits are unstable for $u_0 > 1/(3r_s) \Leftrightarrow r_0 < 3r_s$.

For null geodesics ($k = 0$),

$$u_0 \left(1 - \frac{3}{2} r_s u_0\right) = 0 \quad \Rightarrow \quad u_0 = 0, \ \frac{2}{3r_s}.$$

Then, as before,

$$\delta u'' + \delta u = 3r_s \left(\frac{2}{3r_s}\right) \delta u \quad \Rightarrow \quad \delta u'' - \delta u = 0.$$

This has solution

$$\delta u(\phi) = c_1 \sinh\phi + c_2 \cosh\phi,$$

meaning that there are no stable deviations away from a circular orbit for null geodesics; these will either be plunging or unbound orbits.

ii.) Mercury orbits the sun in an ellipse with semi-latus rectum of approximately 5.546×10^{10} m. Using your results from the previous part, calculate the perihelion advance of Mercury. [*Hint: Find the correction to the orbital period due to General Relativity.*]

Solution: In general, the calculation of the perihelion shift of an orbit must take into account the eccentricity of the orbit. However, in turns out that one obtains the same result by considering perturbations around a *circular* orbit instead. We thus have that

$$T_\phi = \frac{2\pi}{\omega_\phi} = \frac{2\pi}{\sqrt{1 - 3r_s u_0}} \simeq 2\pi \left(1 + \frac{3}{2} r_s u_0\right).$$

Thus,

$$\Delta\phi = 3\pi r_s u_0 = \frac{6\pi GM}{c^2 R},$$

where we have restored units in the last expression. For Mercury, we find that $\Delta\phi \sim 5 \times 10^{-7}$, which is approximately 42.98 seconds of arc per century.

Problem Set 3: Linearised Gravity

14. There is a class of metrics which admit coordinates such that

$$g_{ab} = \eta_{ab} + \phi\, n_a n_b,$$

with n_a satisfying $\eta^{ab} n_a n_b = 0$, where $\eta_{ab} = \eta^{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

i.) By looking for an inverse metric of the form $g^{ab} = \eta^{ab} + \psi\, n^a n^b$, show both that n_a is null with respect to the metric g_{ab}, and that $\psi = -\phi$. [*Hint: Taking the trace may be useful here.*]

Solution: By definition, we have that

$$\begin{aligned} g_{ab} g^{bc} = \delta^a{}_c &= (\eta_{ab} + \phi n_a n_b)(\eta^{bc} + \psi n^b n^c) \\ &= \delta^a{}_c + \phi \eta^{bc} n_a n_b + \psi \eta_{ab} n^b n^c + \phi\psi n_a n_b n^b n^c. \end{aligned}$$

Assuming that we are working within a spacetime of dimension d. Then, taking the trace throughout,

$$d = d + \phi\psi (n_a n^a)^2 \quad \Rightarrow \quad n^a n_a = g^{ab} n_a n_b = 0,$$

meaning that n_a is also null with respect to the metric g_{ab}. Furthermore, this means that the raising and lowering of the indices on n_a can be done with either g_{ab} or η_{ab}. This means that

$$\phi \eta^{bc} n_a n_b + \psi \eta_{ab} n^b n^c = (\phi + \psi) n_a n^c = 0 \quad \Rightarrow \quad \psi = -\phi.$$

ii.) Show that $\Gamma^a{}_{bc} n^b n^c = 0$ and $\Gamma^a{}_{bc} n_a n^b = 0$. Use this to show that if n_a is geodesic with respect to the Minkowski metric, $n_a \eta^{ab} \partial_b n_c = 0$, then it is also geodesic with respect to the curved metric g_{ab}, $n^a \nabla_a n_b = 0$.

Solution: Using the definition of the Christoffel symbols and of the metric, we have that

$$\begin{aligned}\Gamma^a{}_{bc}n^b n^c &= \frac{1}{2}n^b n^c g^{ad}\left[\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right] \\ &= \frac{1}{2}n^b n^c g^{ad}\left[\partial_b(\phi n_d n_c) + \partial_c(\phi n_b n_d) - \partial_d(\phi n_b n_c)\right] \\ &= \frac{1}{2}n^b n^c g^{ad}\left[\phi n_d(\partial_b n_c) + \phi n_d(\partial_c n_b)\right],\end{aligned}$$

where we have used the null condition on n_a. Now, we note that

$$\partial_a\left(n^b n_b\right) = 2(\partial_a n_b)n^b = 0,$$

meaning that n_a contracted into itself vanishes, even if there is a partial derivative acting on one of them. From this, it follows that $\Gamma^a{}_{bc}n^b n^c = 0$, and similarly for $\Gamma^a{}_{bc}n_a n^c = 0$. This means that

$$n^b\nabla_b n_a = n^b\left(\partial_b n_a - \Gamma^c{}_{ab}n_c\right) = n^b\partial_b n_a - \Gamma^c{}_{ab}n^b n_c = 0,$$

meaning that n_a is also geodesic with respect to the curved metric.

iii.) Consider the special case for which

$$\phi = \frac{2GM}{r}, \quad n_a = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right),$$

where $r = \sqrt{x^2+y^2+z^2}$. Using the results of the previous part, show that n_a is geodesic with respect to this metric. Show also that $n_a \mathrm{d}x^a = dx^0 + dr$. Finally, show that the metric in question is actually the Schwarzschild solution (8). [*Hint: Look for a coordinate change of the form* $x^0 = t + \xi(r)$.]

Solution: We have shown that if n_a is geodesic with respect to the Minkowski metric, then it is also geodesic with respect to g_{ab}. Then,

$$n_a \eta^{ab} \partial_b n_c = -n_0 \partial_0 n_c + n_i \partial_i k_c = \frac{x_i}{r} \partial_i \frac{x_j}{r} = \frac{x_j}{r} (\frac{1}{r} - \frac{1}{r^2} (x^j \partial_j r)) = 0,$$

as required. Then, we also have that.

$$n_a \mathrm{d}x^a = n_0 \mathrm{d}x^0 + \frac{x_i}{r} \mathrm{d}x^i = \mathrm{d}x^0 + \mathrm{d}r.$$

Define $\mathrm{d}\Omega_{(2)}$ as the differential solid angle in two-dimensions. Then:

$$g_{ab} \mathrm{d}x^a \mathrm{d}x^b = -(\mathrm{d}x^0)^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{(2)}^2 + \phi((\mathrm{d}x^0)^2 + 2\mathrm{d}x^0 \mathrm{d}r + \mathrm{d}r^2).$$

Looking for a coordinate change of the form $\mathrm{d}x^0 = \mathrm{d}t + \xi \mathrm{d}r$,

$$\begin{aligned} g_{ab} \mathrm{d}x^a \mathrm{d}x^b &= -\mathrm{d}t^2 - \xi^2 \mathrm{d}r^2 - 2\xi \mathrm{d}t\mathrm{d}r + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{(2)}^2 \\ &\quad + \phi(\mathrm{d}t^2 + \xi^2 \mathrm{d}r^2 + 2(\xi')\mathrm{d}t\mathrm{d}r + 2(\mathrm{d}t + \xi \mathrm{d}r)\mathrm{d}r + \mathrm{d}r^2) \\ &= -(1-\phi)\mathrm{d}t^2 + (1 + \phi(1+\xi)^2 - \xi^2)\mathrm{d}r^2 \\ &\quad + 2(\phi(1+\xi) - \xi)\mathrm{d}t\mathrm{d}r + r^2 \mathrm{d}\Omega_{(2)}^2. \end{aligned}$$

Demanding that the metric be diagonal gives $\xi = \phi/(1-\phi)$, such that

$$(1 + \phi(1+\xi)^2 - \xi^2) = 1 + \xi = \frac{1}{1-\phi},$$

meaning that

$$\mathrm{d}s^2 = -(1-\phi)\mathrm{d}t^2 + (1-\phi)^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{(2)}^2,$$

which is the Schwarzschild solution (8) for $\phi = 2GM/r$.

Throughout the remainder of this problem set, we will be working within the weak gravity limit, for which

$$g_{ab} = \eta_{ab} + h_{ab}, \quad g^{ab} = \eta^{ab} - h^{ab}, \quad |h_{ab}| \ll 1, \tag{11}$$

That is, the metric consists of a small perturbation on a Minkowski background.

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15. In this question, we shall fix the constant c_1 in Einstein's field equations

$$G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R = c_1 T^{ab}$$

where G^{ab} is the Einstein tensor, R^{ab} is the Ricci tensor, R the Ricci scalar, and T^{ab} the stress-energy tensor. Confirm that the covariant derivative of the left-hand side of this expression vanishes. What physical condition does this express?

Solution: Starting from the Bianchi identity

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0,$$

we raise a and contract it with c, such that

$$R_{bd;e} + R^{c}{}_{bde;c} - R_{be;d} = 0,$$

where we have made use of the antisymmetry in the first two indices in the last term. Again, raise b and contract it with d, making use of the symmetries of R_{abcd}:

$$R_{;e} - R^{c}{}_{e;e} - R^{d}{}_{e;d} = R_{;e} - 2R^{c}{}_{e;c} = (\delta^{c}{}_{e}R - 2R^{c}{}_{e})_{;e} = 0.$$

Now, the metricity condition implies that $g^{de}{}_{;c} = 0$, such that

$$g^{de}\left(\delta^{c}{}_{e}R - 2R^{c}{}_{e}\right)_{;e} = \left(g^{cd} - R - 2R^{cd}\right)_{;c} = 0.$$

Relabelling indices, this becomes

$$\left(R^{ab} - \frac{1}{2}g^{ab}R\right)_{;a} = 0.$$

This means that the covariant derivative of the left-hand side of the field equations vanishes. This is a requirement for energy conservation.

Find an expression for the geodesic equation (5) in the weak-gravity limit (11). By comparing it with the Newtonian limit

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\nabla\Phi,$$

show that $h_{00} = -2\Phi$.

Solution: In the weak-gravity limit (11), we assume that $\dot{x}^a \simeq \dot{x}^0 \gg \dot{x}^i$, such that we can write the geodesic equation (5) as

$$\ddot{x}^a + \Gamma^a{}_{00}(\dot{x}^0)^2 = \ddot{x}^a + \Gamma^a{}_{00} = 0.$$

Then,

$$\Gamma^a{}_{00} = -\frac{1}{2}g^{ac}\partial_c g_{00} \simeq -\frac{1}{2}\eta^{ac}\partial_c g_{00} \simeq -\frac{1}{2}\partial_a g_{00}$$

where the last equality follows from the fact that we assume that the metric is slowly varying in time. For timelike geodesics,

$$\ddot{x}^a = \frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^a}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} + \mathcal{O}(h_{ab}).$$

Putting this together, we have that

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} = \frac{1}{2}\partial_a h_{00}.$$

Comparison of the spatial component of this with the Newtonian limit clearly yields $h_{00} = -2\Phi$.

Recalling the Riemann tensor (7), show that the Ricci tensor is given by

$$R_{bd} = R^a{}_{bad} = \frac{1}{2}\left(\partial_d\partial^a h_{ab} + \partial_b\partial^a h_{da} - \partial_a\partial^a h_{bd} - \partial_b\partial_d h\right) \tag{12}$$

where $h = h^a{}_a$. Show further that the coordinate gauge transformation $h_{ab} \mapsto h_{ab} + \partial_a\xi_b + \partial_b\xi_a$ leaves (12) unchanged. Adopting the *harmonic gauge* condition

$$\partial^a \bar{h}_{ab} = \partial^a\left(h_{ab} - \frac{1}{2}\eta_{ab}h\right) = 0,$$

show that

$$(\partial^a\partial_a)\bar{h}_{bd} = -2c_1 T_{bd}. \tag{13}$$

Comparing the timelike component of (13) to the Newtonian limit $\nabla^2\Phi = 4\pi G\rho$, show that $c_1 = 8\pi G$. Restoring units, this becomes $8\pi G/c^4$, the familiar result.

Solution: Recalling the definition of the Christoffel symbols, it is easy to see that

$$\Gamma^a{}_{bc} = \frac{1}{2}\eta^{ad}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right) + \mathcal{O}(h_{ab}^2).$$

In the Riemann tensor, we ignore products of the Christoffel symbols, since these are already of $\mathcal{O}(h_{ab}^2)$. Then, it follows that

$$\begin{aligned} R^a{}_{bcd} &= \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{cb} \\ &= \frac{1}{2}\eta^{ae}\left(\partial_b\partial_c h_{de} + \partial_d\partial_e h_{cb} - \partial_c\partial_e h_{bd} - \partial_d\partial_b h_{ce}\right). \end{aligned}$$

Contracting a with c, the desired result follows.

Under the coordinate gauge transformation, we have that

$$\begin{aligned} &\partial_d\partial^a(h_{ab} + \partial_a\xi_b + \partial_b\xi_a) + \partial_b\partial^a(h_{da} + \partial_d\xi_a + \partial_a\xi_d) \\ &\qquad - \partial^a\partial_a(h_{bd} + \partial_b\xi_d + \partial_d\xi_b) - \partial_b\partial_d(h + 2\partial^a\xi_a), \\ &= R_{bd} + \partial^a\partial_a\partial_d\xi_b + 2\partial^a\partial_b\partial_d\xi_a + \partial^a\partial_a\partial_b\xi_d - \partial^a\partial_a\partial_b\xi_d - \partial^a\partial_a\partial_d\xi_b - 2\partial^a\partial_b\partial_d\xi_a, \\ &= R_{bd}, \end{aligned}$$

as required. It is helpful to re-write our expression for R_{bd} in terms of $\bar{h}_{ab}$, vis.:

$$\begin{aligned} R_{bd} &= \frac{1}{2}\left[\partial_b\left(\partial^a h_{ad} - \frac{1}{2}\partial_d h\right) + \partial_d\left(\partial^a h_{ab} - \frac{1}{2}\partial_b h\right) - \partial_a\partial^a h_{bd}\right], \\ &= \frac{1}{2}\left[\partial^a\partial_b\left(h_{ad} - \frac{1}{2}\eta_{ad}h\right) + \partial^a\partial_b\left(h_{ab} - \frac{1}{2}\eta_{ab}h\right) - \partial^a\partial_a h_{bd}\right], \\ &= \frac{1}{2}\left[\partial^a\partial_b\bar{h}_{ad} + \partial^a\partial_d\bar{h}_{ab} - \partial^a\partial_a h_{bd}\right]. \end{aligned}$$

Similarly, the Ricci scalar becomes

$$R = \eta^{bd}R_{bd} = \frac{1}{2}\left[\partial^a\partial^b h_{ab} + \partial^a\partial^b h_{ab} - \partial^a\partial_a h\right] = \partial^a\partial^b\bar{h}_{ab} - \frac{1}{2}\partial^a\partial_a h.$$

Thus, we can finally write the Einstein tensor as

$$G_{bd} = R_{bd} - \frac{1}{2}\eta_{bd}R = \frac{1}{2}\left[\partial^a\partial_d\bar{h}_{ab} + \partial^a\partial_b\bar{h}_{ad} - \partial^a\partial_a\bar{h}_{bd} - \eta_{bd}\partial^a\partial^c\bar{h}_{ac}\right].$$

Adopting the harmonic gauge, this clearly reduces to $G_{bd} = -\frac{1}{2}\partial^a\partial_a\bar{h}_{bd}$, giving (13). The timelike component of this equation is

$$(\partial^a\partial_a)\bar{h}_{00} = \partial^a\partial_a\left(h_{00} + \frac{1}{2}h\right) = -4(\partial^a\partial_a)\Phi = -2c_1\rho \quad \Rightarrow \quad c_1 = 8\pi G,$$

upon comparison with the Newtonian limit. Here, we have used the fact that $T_{ij} \simeq 0$ in the weak-gravity limit to write that

$$\bar{h}_{ij} \simeq 0 \quad \Rightarrow \quad h_{ij} = \frac{1}{2}\eta_{ij}h \quad \Rightarrow \quad h = -h_{00} + h_{ii} = 2h_{00}.$$

16. We shall now consider vacuum solutions to the gravitational wave equation (13), vis.:

$$(\partial^c \partial_c)\bar{h}_{ab} = 0.$$

We seek plane-wave solutions of the form $\bar{h}_{ab} = \chi_{ab} \exp[ik_c x^c]$, for wavevector $k^a = (\omega, \mathbf{k})$ and a constant, symmetric tensor χ_{ab}.

i.) Show that solutions of such form propagate at the speed of light.

Solution: Substituting the plane wave solution into the wave equation, we find that

$$-k_c k^c \bar{h}_{ab} = 0 \quad \Rightarrow \quad k_c k^c = 0.$$

This means that the solutions propagate along null geodesics (at the speed of light).

ii.) Show that the wavevector k^a is orthogonal to χ_{ab}.

Solution: The harmonic gauge condition implies that

$$\partial^a \bar{h}_{ab} = ik^a \chi_{ab} e^{ik_c x^c} = 0 \quad \Rightarrow \quad k^a \chi_{ab} = 0.$$

This means that the wavevector is orthogonal to χ_{ab}, up to second order in the metric perturbation.

iii.) Write down the conditions for the metric perturbation to be purely spatial and traceless; perturbations satisfying these conditions are said to be in the *transverse-traceless (TT)* gauge. Show that this implies $\partial^i h_{ij} = 0$.

Solution: For the perturbation to be purely spatial, we require that $\chi_{a0} = \chi_{0a} = 0$, as this eliminates all mixed components. The condition for it to be traceless is $\chi^a{}_a = 0$. Then, the harmonic gauge condition tell us that

$$\partial^a \bar{h}_{ab} = \partial^0 \left(h_{0b} - \frac{1}{2}\eta_{0b} h \right) + \partial^i \left(h_{ib} - \frac{1}{2}\eta_{ib} h \right) = \partial^i h_{ij} = 0$$

since $h = 0$ by the condition $\chi^a{}_a = 0$.

iv.) Finally, write down the most general form of χ_{ab} for a perturbation with wavevector $k^a = (\omega, 0, 0, \omega)$.

Solution: The combination of $k^a = (\omega, 0, 0, \omega)$ and the condition $k^a \chi_{ab} = 0$ tells us that $\chi_{zb} = 0$. Then, given that χ_{ab} is both traceless and symmetric, the most general form is

$$\chi_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \chi_{11} & \chi_{12} & 0 \\ 0 & \chi_{12} & -\chi_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This means that the wave is entirely characterised by the components χ_{11} and χ_{12}.

17. With some algebra, one can show from (13) that the spatial components of the metric perturbation $\bar{h}_{ij}$ at some field event (ct, r) in response to a source event $(ct_s, \mathbf{r}_s)$ evolve according to the *quadrupole formula*

$$\bar{h}_{ij} = \frac{2G}{c^4}\frac{\ddot{I}_{ij}}{r}, \quad I_{ij} = \frac{1}{c^2}\int \mathrm{d}^3\mathbf{r}_s\, r_s^i r_s^j\, T^{00}(t_s, \mathbf{r}_s), \tag{14}$$

where we have restored units for the sake of clarity. Note that the time derivatives are taken with respect to the retarded time $t_s = t - r/c$ of the source event. One can also show that the gravitational luminosity of the source is given by

$$L_{\mathrm{GW}} = \frac{G}{5c^5}\dddot{J}_{ij}\dddot{J}^{ij}, \quad J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{mn}I_{mn}. \tag{15}$$

Hence, given a particular (time-dependent) distribution of matter T^{00}, we can find the local perturbation away from the Minkowski metric, and the resultant observed gravitational luminosity. Indeed, (15) was useful in inferring the properties of the black holes in the famous gravitational wave observation GW150914 by LIGO/Virgo.

In this question, we consider a black hole binary system merger, wherein two black-holes of masses m_1 and m_2 orbiting at r_1 and r_2 relative to the origin gradually spiral in towards one-another. We shall assume that both black holes remain on circular orbits during the merger.

i.) Assuming that the orbital motion of the bodies can be confined to the equatorial $(\theta = \pi/2)$ plane, write down an expression for the time dependent mass-density of the bodies in terms of m_1, m_2, r_1, r_2 and relevant spatial coordinates. Using this, show that

$$I_{xx} = \mu r^2 \cos^2\phi, \quad I_{yy} = \mu r^2 \sin^2\phi, \quad I_{xy} = I_{yx} = \mu r^2 \sin\phi\cos\phi,$$

where ϕ is the angular coordinate in the orbital plane, and $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the system.

Solution: The mass density can be expressed as

$$\rho = \delta(z)\left[m_1\delta(x - r_1\cos\phi)\delta(y - r_1\sin\phi) + m_2\delta(x + r_2\cos\phi) + \delta(y + r_2\sin\phi)\right].$$

Moving to centre of mass coordinates, we have that

$$r_1 = \frac{\mu}{m_1}r, \quad r_2 = \frac{\mu}{m_2}r, \quad \mu = \frac{m_1 m_2}{m_1 + m_2},$$

meaning that we can calculate the non-vanishing moments of the quadrupole tensor

$$I_{xx} = \int \mathrm{d}^3\mathbf{r}\,\rho x^2 = m_1 r_1^2\cos^2\phi + m_2 r_2^2\cos^2\phi = \mu r^2\cos\phi,$$

and similarly for I_{yy}, I_{xy} and I_{yx}.

ii.) Assuming that $\phi = \omega t$, give an expression for the orbital frequency ω in terms of r and other constants. Using the quadrupole formula (15), show that the gravitational luminosity of the binary is given by

$$L_{\rm GW} = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 (m_1+m_2)}{r^5},$$

where r is the radius of motion of the centre of mass. What 'dodgy' assumption has been made in this derivation? [*Hint: your results of question 11 in the previous problem set may be useful here.*]

Solution: In question 11, we showed that the frequency for a circular orbit around a point-mass is given by

$$\omega(t) = \sqrt{\frac{GM}{r(t)^3}} = \sqrt{\frac{G(m_1+m_2)}{r(t)^3}}.$$

It is then straightforward to calculate the trace-reversed quadrupole tensor from the quadrupole moments listed above:

$$J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I_{kk} = \frac{1}{2}\mu r^2 \begin{pmatrix} \cos 2\omega t + 1/3 & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t + 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix}.$$

We take time derivatives of this expression assuming that the frequency $\omega(t)$ is a slow function of time; this is the 'dodgy' assumption referred to in the question, as we will see that $\mathrm{d}r/\mathrm{d}t$ diverges as $r \to 0$, as one would expect. This makes this classical model of the merger not a very good approximation to the real system, as the majority of the gravitational wave radiation is released during times for which $r \ll r_0$. In any case, taking the aforementioned time derivatives yields

$$\dddot{J}_{ij} = 4\mu r^2 \omega^3 \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

such that

$$\dddot{J}_{ij}\dddot{J}^{ij} = 16\mu^2 r^4 \omega^6 \mathrm{Tr} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 32\mu^2 r^4 \omega^6.$$

Then, the gravitational luminosity of the binary is given by (15):

$$L_{\rm GW} = \frac{32}{5}\frac{G}{c^5}\mu^2 r^4 \omega^6 = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 (m_1+m_2)}{r^5}.$$

iii.) Using the virial theorem, or otherwise, find an expression for the total energy of the binary in terms of r and other constants.

Solution: The virial theorem ($2\langle K\rangle = n\langle V\rangle$ for $V(r) = \alpha r^n$) tells us that the total energy of the binary is given by

$$E_{\text{tot}} = \langle K\rangle + \langle V\rangle = \frac{1}{2}\langle V\rangle = \frac{1}{2}\frac{Gm_1m_2}{r}.$$

iv.) Hence, for a given initial radius r_0, show that the time taken for the black holes to merge is given by

$$t_{\text{merge}} = \frac{5}{256}\frac{c^5}{G^3}\frac{r_0^4}{m_1m_2(m_1+m_2)}.$$

Does this expression scale how you would expect? Find the time taken for two black holes with equal masses $m_1 = m_2 = 60M_\odot$ initially located at one astronomical unit from one another. Is your answer reasonable?

Solution: By energy conservation, the energy lost due to gravitational waves decreases the total energy of the binary. Thus,

$$\frac{\mathrm{d}E_{\text{tot}}}{\mathrm{d}t} = \frac{1}{2}Gm_1m_2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r}\right) = L_{\text{GW}} \quad\Rightarrow\quad \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{64}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{r^3}.$$

Given the initial condition $r(0) = r_0$, this can be solved for

$$r^4 = r_0^4 - \frac{256}{5}\frac{G^3}{c^5}m_1m_2(m_1+m_2)t.$$

Letting $r = 0$, we find the desired expression for t_{merge}. This scales as one would expect; the greater the initial separation, the larger the time until the merger, but the greater the mass, the shower the time as more energy is lost due to gravitational radiation. For the masses and initial separation given, we find that $t_{\text{merge}} \sim 10^{19}$ s. This answer is not particularly reasonable, as it is approximately 50 times the age of the universe, even though we have observed black hole mergers in finite time.

18. *When two black holes of masses m_1 and m_2 collide to form a single large black hole of mass M, the total area of the horizon must increase.*

By considering radial, null geodesics in Scwharzschild spacetime (8), justify this statement by invoking causality. [*Hint: Think about light-cones; how are they orientated for $r < r_s$?*] Then, find an expression for an upper bound on the total energy that can be released during the merger. Find a value for this upper bound for $m_1 = m_2 = 60M_\odot$, and confirm that this is larger than the total energy emitted due to gravitational waves during the binary merger studied in question 17.

Solution: In Schwarzschild spacetime (8), all timelike and null paths for $r < r_s$ are causally bounded, meaning that r_s is an event horizon for the black hole (the light cones are inwardly pointing for $r < r_s$). This means that points inside the Schwarzschild radii of the original two black holes cannot lie outside the Schwarzschild radius of the final black hole, as otherwise this would correspond to the propagation of something outwards across the event horizon. This means that the total area bounded by the event horizon must increase.

Given the spherical symmetry of (8), the surface area element is given by $\mathrm{d}S = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ as one would expect. From this, it is clear that the area bounded by a given r_s is $A = 4\pi r_s^2$. This means that

$$M^2 - m_1^2 - m_2^2 \geq 0 \quad \Rightarrow \quad M \geq \sqrt{m_1^2 + m_2^2}.$$

By mass-energy equivalence, it follows that the energy that can be emitted during a black-hole merger is bounded from above by

$$\Delta E_{\mathrm{GW}} \leq m_1 + m_2 - \sqrt{m_1^2 + m_2^2}.$$

For the given masses, $\Delta E_{\mathrm{GW}} \sim 10^{48}$ J. Now, the total initial energy of the binary is given by $E_{\mathrm{tot}} = G m_1 m_2/(2r_0) \sim 10^{42}$ J. Even if all of the energy of the binary system where released as gravitational waves, this would still be below the bound provided by ΔE_{GW}.

19. Consider two point masses m located at $(\ell/2, 0, 0)$ and $(-\ell/2, 0, 0)$ respectively that are constrained to move along the x-axis. These are impinged upon by a gravitational wave travelling along $\hat{\mathbf{z}}$, with metric perturbation satisfying $h_{xx} = -h_{yy} = A_{xx}\cos(kz - \omega t)$. Find an expression for the proper distance of each of the masses from the origin as a function of time, to first order in the metric perturbation. Using (15), show that the time-averaged gravitational luminosity of the particle response is given by

$$\langle L_{\mathrm{GW}} \rangle_t = \frac{G}{60c^5} m^2 \omega^6 \ell^4 A_{xx}^2.$$

Solution: The mass distribution of the two masses is given by

$$\rho(x,t) = m\delta(x - x_1(t)) + m\delta(x + x_1(t)),$$

where the position of the masses is given by the proper distance

$$\begin{aligned} x_1(t) &= \int_0^{\ell/2} \mathrm{d}s = \int_0^{\ell/2} \sqrt{-g_{ab}\mathrm{d}x^a\mathrm{d}x^b} \\ &= \int_0^{\ell/2} \mathrm{d}x \sqrt{1 + h_{xx}(t,0)} \simeq \frac{\ell}{2}\left(1 + \frac{1}{2}h_{xx}(t,0)\right). \end{aligned}$$

Then,

$$I_{xx} = \int \mathrm{d}^3\mathbf{r}\, \rho x^2 = 2mx_1^2 \simeq \frac{1}{2}m\ell^2(1 + h_{xx}) = \frac{1}{2}m\ell^2(1 + A_{xx}\cos(\omega t)).$$

Using this, it is easy to show that

$$\dddot{J}_{ij} = \dddot{I}_{ij} - \frac{1}{3}\delta_{ij}\dddot{I}_{xx} = \frac{1}{6}m\omega^3\ell^2 A_{xx}\sin(\omega t)\mathrm{diag}(2,-1,-1).$$

Then,

$$\langle L_{\mathrm{GW}}\rangle_t = \frac{G}{5c^5}\left\langle \dddot{J}_{ij}\dddot{J}^{ij}\right\rangle_t = \frac{G}{30c^5}m^2\omega^6\ell^4 A_{xx}^2\left\langle \sin^2(\omega t)\right\rangle_t = \frac{G}{60c^5}m^2\omega^6\ell^4 A_{xx}^2.$$

The energy flux due to gravitational radiation is given by

$$F^i = -\frac{c^4}{32\pi G}(\partial_i \bar{h}^{ab}\partial_t \bar{h}_{ab}).$$

The cross-section σ_{GW} for gravitational interaction is defined to be the ratio of the average luminosity to the average incoming flux. Why is this a good definition of the cross-section? Show that

$$\sigma_{\mathrm{GW}} = \frac{2\pi}{15}r_s^2\left(\frac{\omega\ell}{c}\right)^4, \quad r_s = \frac{2Gm}{c^2}.$$

Give a physical interpretation of the factor $(\omega\ell/c)$. Evaluate this numerically for $m = 10\,\mathrm{kg}$, $\ell = 10\,\mathrm{m}$ and $\omega = 20\,\mathrm{rad\,s^{-1}}$ and compare this with the typical weak interaction cross section of $10^{-48}\,\mathrm{m}^2$. Hence justify the statement: *Gravity is the weakest force.*

Solution: This is a good definition of the cross-section as it is essentially the ratio of the response to the forcing for a gravitational wave. It will also be independent of the amplitude of the oscillations A_{xx}. Then:

$$\partial_i \bar{h}^{ab} = -kA_{xx}\sin(kz - \omega t), \quad \partial_t \bar{h}_{ab} = A_{xx}\omega \sin(kz - \omega t),$$

meaning that $(k = \omega/c)$

$$\langle F^z \rangle_t = \frac{c^3\omega^2 A_{xx}^2}{16\pi G} \left\langle \sin^2(\omega t) \right\rangle_t = \frac{c^3\omega^2 A_{xx}^2}{32\pi G}$$

Thus, the cross-section is given by

$$\sigma_{\rm GW} = \frac{\langle L_{\rm GW} \rangle_t}{\langle F^z \rangle_t} = \frac{8\pi G^2 m^2 \omega^4 \ell^4}{15c^8} = \frac{2\pi}{15} r_s^2 \left(\frac{\omega \ell}{c}\right)^4,$$

as required. The factor $(\omega\ell/c)$ is essentially the ratio of the effective 'speed' for the oscillatory response to the speed of light, which we expect to be small. Evaluating this for the given values, we find that $\sigma_{\rm GW} \sim 10^{-77}$ m^2. This is almost thirty orders of magnitude smaller than the typical weak interaction cross section, giving justification to the statement.

Problem Set 4: Cosmology

20. The *Friedmann-Robertson-Walker (FRW) metric*

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right], \tag{16}$$

is a solution to Einstein's field equations over a three-dimensional manifold of constant curvature.

i.) What key assumptions about the nature of our universe are used in the derivation of (16)? Give physical interpretations of the coordinate t, the function $a(t)$ and the constant k.

Solution: The two key assumptions about the nature of our universe used to derive the FRW metric (16) are:

- Homogeneity - That, at any given time, the universe looks the same at every point in space.
- Isotropy - That, at any given time, the universe looks the same in any direction of observation.

The coordinate time t corresponds to the proper time of isotropic observers. The function $a(t)$ is the *scale factor* that parametrises the relative expansion of the universe. Lastly, we note that the rescaling $k \mapsto k/|k|$, $r \mapsto \sqrt{|k|}r$, $a \mapsto a/\sqrt{|k|}$ leaves (16) unchanged. This means that the only relevant parameter is $\mathrm{sgn}(k)$, for which there are three solutions:

- $k = -1$, corresponding to constant negative curvature (open).Fr
- $k = 0$, corresponding to no curvature (flat).
- $k = 1$, corresponding to constant positive curvature (closed).

ii.) Consider two observers located at some fixed comoving distance ℓ in flat spacetime. Suppose that one observer emits a photon of wavelength λ at time t, which is observed by a second observer as λ_0 at time t_0. Show that the cosmological redshift factor z can be written as

$$z = \frac{\lambda_0}{\lambda} - 1,$$

and find an expression for z in terms of a, t and t_0.

Solution: Considering null, radial geodesics in FRW spacetime (16), we have that

$$\mathrm{d}t = \pm \frac{a(t)\,\mathrm{d}r}{\sqrt{1-kr^2}}.$$

Then, the comoving distance in flat spacetime is given by

$$\ell = \int_t^{t_0} \frac{\mathrm{d}t'}{a(t')} = \int_{t+\Delta t}^{t_0+\Delta t_0} \frac{\mathrm{d}t'}{a(t')},$$

since the comoving distance is time-independent as it is associated with the conformal coordinates. Then, we note that

$$\int_{t+\Delta t}^{t_0+\Delta t_0} = \int_t^{t_0} + \int_{t_0}^{t_0+\Delta t_0} - \int_t^{t+\Delta t}$$

meaning that

$$\int_{t_0}^{t_0+\Delta t_0} \frac{\mathrm{d}t'}{a(t')} = \int_t^{t+\Delta t} \frac{\mathrm{d}t'}{a(t')} \quad \Rightarrow \quad \frac{\Delta t_0}{a(t_0)} = \frac{\Delta t}{a(t)},$$

where we have assumed that $\Delta t_0 \ll t_0$ and $\Delta t \ll t$. Thus, it follows that

$$1 + z = \frac{a(t_0)}{a(t)} = \frac{\lambda_0}{\lambda}.$$

This can be re-arranged for the desired expression.

iii.) Find expressions for the *Hubble constant* H_0 and the *deceleration parameter* q_0 in the expansion

$$\frac{a(t)}{a(t_0)} = 1 - H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \dots,$$

Then, show that for small z the comoving distance ℓ can be expressed as

$$\ell = \frac{1}{H_0}\left[z - \frac{1}{2}z^2(1+q_0) + \dots\right].$$

Solution: We can expand the scale factor around the current time t_0 as

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a}(t_0)(t - t_0)^2 + \dots$$

Thus, we have that

$$\frac{1}{1+z} = \frac{a(t)}{a(t_0)} = 1 + \frac{\dot{a}(t_0)}{a(t_0)}(t - t_0) + \frac{1}{2}\frac{\ddot{a}(t_0)}{a(t_0)}(t - t_0)^2 + \dots$$

Comparing this with the given expression, we find that

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{\dot{a}_0}{a_0}, \quad q_0 = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}(t_0)^2} = -\frac{a_0\ddot{a}_0}{\dot{a}_0^2}.$$

Now, we can invert the given expression

$$H_0(t - t_0) = z + \alpha z^2 + \dots = z - \left(1 + \frac{1}{2}q_0\right) z^2 + \dots$$

Then comoving distance can similarly be expanded as

$$\ell = \int \frac{\mathrm{d}t'}{a(t')} = \frac{1}{a_0}\left[(t - t_0) + \frac{1}{2}H_0(t - t_0)^2 + \dots\right] = \frac{1}{H_0}\left[z - \frac{1}{2}z^2(1 + q_0) + \dots\right],$$

as required.

21. Consider an isotropic metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

Why are there only four non-zero components of the Ricci tensor R_{ab}? Show explicitly that

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2k)\gamma_{ij},$$

in the case of the FRW metric (16). Here, the dot denotes differentiation with respect to the coordinate time t. Write down the stress-energy tensor for a perfect fluid with no overall velocity. Hence, show that Einstein's field equations in the presence of a cosmological constant Λ

$$G_{ab} = 8\pi G T_{ab} - \Lambda g_{ab},$$

reduce to the *Friedmann equations*:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{17}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{18}$$

Solution: There are only four non-zero components of R_{ab} due to isotropy. In order to calculate the components of the Ricci tensor, we first need the relevant Christoffel symbols. For this, it is useful to use following identities for the Christoffel symbols for a symmetric metric g_{ab}:

$$\Gamma^a{}_{ba} = \frac{1}{2} g^{aa} \partial_b g_{aa} \quad (a = b \text{ permitted}),$$
$$\Gamma^a{}_{bb} = -\frac{1}{2} g^{aa} \partial_a g_{bb} \quad (a \neq b),$$
$$\Gamma^a{}_{bc} = 0 \quad (a, b, c \text{ distinct}).$$

Then, we have that

$$\Gamma^0{}_{ij} = a\dot{a}\gamma_{ij}, \quad \Gamma^i{}_{0i} = \frac{\dot{a}}{a}, \quad \Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r},$$
$$\Gamma^r{}_{\theta\theta} = -r(1-kr^2), \quad \Gamma^r{}_{\phi\phi} = -r(1-kr^2)\sin^2\theta, \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi{}_{\theta\phi} = \cot\theta.$$

Using these, one can show the desired expressions for R_{00} and R_{ij}. It is sufficient to do a single spatial component, as the others follow by isotropy.

The stress-energy tensor of a fluid with no overall velocity is given by

$$T_{ab} = pg_{ab} + (\rho + p)u_a u_b, \quad u_a = (1, 0) \quad \Rightarrow \quad T_{ab} = pg_{ab} + (\rho + p)\delta_{00}.$$

Using our previous results, the Ricci scalar is

$$R = \frac{6}{a^2}(\ddot{a}a + \dot{a}^2 + k),$$

while the components of the Einstein tensor are

$$G_{00} = R_{00} + \frac{1}{2}R = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2}$$
$$G_{ij} = R_{ij} - \frac{1}{2}a^2\gamma_{ij}R = -(2a\ddot{a} + \dot{a}^2 + k)\gamma_{ij}.$$

Using the expression for the stress-energy tensor from above, $T_{00} = \rho$ and $T_{ij} = a^2 p\gamma_{ij}$. Then, the timelike and spatial components of Einstein's field equation are

$$3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} = 8\pi G\rho + \Lambda, \quad 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi Gp + \Lambda$$

respectively. The first of these expressions is (17), while (18) follows from substituting the first expression into the second.

22. By considering (17) and (18), show that the mass density ρ satisfies the continuity equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0.$$

Show that this equation is also a consequence of stress-energy conservation. By adopting the equation of state for a polytropic fluid $p = w\rho$, find how the density depends on the scale factor for general w. Consider the cases of pressureless matter ($w = 0$), radiation ($w = 1/3$) and vacuum energy ($w = -1$), and give physical explanations for the dependence of each on the scale factor.

Solution: Take the time derivative of the first Friedmann equation (17), we have that

$$2\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = \frac{8\pi G}{3}\dot{\rho} + \frac{2k}{a^3}\dot{a}.$$

Substituting the first and second Friedmann equations (17) and (18) into this expression, it is a line of algebra to show the desired result. When considering stress-energy conservation, it is useful to write the stress-energy tensor as

$$T^a{}_b = g^{ac}T_{cb} = \mathrm{diag}(-\rho, p, p, p).$$

Then, stress-energy conservation implies that

$$0 = \nabla_a T^a{}_0 = \partial_a T^a{}_0 + \Gamma^a{}_{a0}T^0{}_0 - \Gamma^b{}_{a0}T^a{}_b = -\partial_0\rho - 3\frac{\dot{a}}{a}(\rho + p),$$

where we have used the Christoffel symbols calculated in the previous question. Letting $p = w\rho$, we find that $\rho \propto a^{-3(1+w)}$. We consider various values of w:

- Pressureless matter ($w = 0$): $\rho_m \propto a^{-3}$. This is simply the decrease in the number density of the particles/matter as the universe is expanding with the scale factor.

- Radiation ($w = 1/3$): $\rho_\gamma \propto a^{-4}$. The energy density of radiation falls off more quickly than matter. This is because the number density of photons decreases in the same way as the number density of pressureless matter, but individual photons also lose energy as a^{-1} due to cosmological redshift.

- Vacuum energy ($w = -1$): $\rho_\Lambda =$ constant. We can write the right-hand side of Einstein's field equations as

  $$8\pi G T_{ab} - \Lambda g_{ab} = 8\pi G\left[\bar{p}g_{ab} + (\bar{\rho} + \bar{p})u_a u_b\right],$$

  where we have defined the effective pressure and density

  $$\bar{p} = p - \frac{\Lambda}{8\pi G}, \quad \bar{\rho} = \rho + \frac{\Lambda}{8\pi G}.$$

  Thus, the effect of the cosmological constant is to decrease the spatial pressure but increase the energy density; this corresponds to the energy density of the vacuum.

We define the *critical density* ρ_c to be the density for which $k = 0$. Find an expression for ρ_c in terms of the Hubble parameter H. Show that the first Friedmann equation (17) can be written as

$$\left(\frac{H}{H_0}\right)^2 = \left[\Omega_\gamma \left(\frac{a_0}{a}\right)^4 + \Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_k \left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda\right], \tag{19}$$

where $a_0 = a(t_0)$ is the scale factor at the current time. Give expressions for the *density ratios* Ω_γ, Ω_m, Ω_k and Ω_Λ. Lastly, show that the second Freidmann equation (18) evaluated at the current time can be written as

$$q_0 = \frac{1}{2}\sum_i (1 + 3w_i)\Omega_i.$$

In a universe consisting of only vacuum energy, is the expansion of the universe accelerating or decelerating?

Solution: Letting $k = 0$ in (17), we have that

$$\rho_c = \frac{3H^2}{8\pi G}.$$

Evaluating this at the current time t_0, and defining the density ratios at current time,

$$\Omega_\gamma = \frac{\rho_\gamma}{\rho_c}, \quad \Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2} = \frac{1}{\rho_c}\frac{\Lambda}{8\pi G}, \quad \Omega_k = -\frac{k}{H_0^2 a_0^2} = -\frac{1}{\rho_c}\frac{3k}{8\pi G a_0^2},$$

(19) follows directly from (17). Recalling the definition of the deceleration parameter from question 20, we have that

$$q_0 = -\frac{a_0 \ddot{a}_0}{\dot{a}_0^2} = -\frac{\ddot{a}_0}{a_0}\left(\frac{\dot{a}}{a_0}\right)^2 = -\frac{\ddot{a}}{a_0} H_0^2.$$

From (18) evaluated at the current time t_0, we arrive at the desired result:

$$q_0 = \frac{1}{2\rho_c}(\rho + 3p) - \frac{\Lambda}{3H_0^2} = \frac{1}{2}\sum_i (1 + 3w_i)\,\Omega_i,$$

For a universe consisting of only vacuum energy, this reduces to $q_0 = -\Omega_\gamma < 0$. This means that the expansion of the universe is accelerating.

23. By introducing the *conformal time* $\mathrm{d}\eta = \mathrm{d}t/a$, show that the *FRW* metric (16) for $k = 0$ can be written as

$$\mathrm{d}s^2 = a(t)^2\left(-\mathrm{d}\eta^2 + dx^2 + dy^2 + dz^2\right).$$

The metric is said to be *conformally flat* on some subset of the overall space. Given that $a^2 > 0$, what is the condition for two events to be connected by a null geodesic in FRW spacetime?

Solution: For $k = 0$, it is trivial to use the fact that $\mathrm{d}t = a d\eta$ to write the metric in the desired form. This means that in order for two events to be connected by a null geodesic in FRW spacetime, they also need to be connected by a null geodesic in flat spacetime with coordinates $\{\eta, x, y, z\}$. This means that η can be used to find the existence of a cosmological horizon; if η is finite, for a (semi-)infinite domain in time, then clearly only a subset of FRW spacetime can be connected by a null geodesic.

Consider a universe containing pressureless matter and radiation. Show that such a cosmological model has a past, but not a future, horizon. If $a(t_0) = 1$, show that the conformal time at present is given by

$$\eta_0 = \int_0^{t_0} d\eta = 2\sqrt{\frac{8\pi G}{3\rho_m}} \left(\sqrt{1 + a_{\mathrm{eq}}} - \sqrt{a_{\mathrm{eq}}}\right),$$

and give an expression for a_{eq}. What is it's physical interpretation?

Solution: We set $\Omega_k = \Omega_\Lambda = 0$ in (19), such that

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega_\gamma a^{-4} + \Omega_m a^{-3}\right],$$

where we have set $a_0 = a(t_0) = 1$. From the definition of conformal time, we have that

$$\int \mathrm{d}\eta = \frac{1}{H_0 \Omega_m^{1/2}} \int \mathrm{d}a \, \frac{1}{\sqrt{a + \Omega_\gamma/\Omega_m}}$$

The conformal time at present is thus

$$\eta_0 = \int_0^{t_0} \mathrm{d}\eta = \frac{1}{H_0 \Omega_m^{1/2}} \int_0^1 \mathrm{d}a \, \frac{1}{\sqrt{a + \Omega_\gamma/\Omega_m}} = 2\sqrt{\frac{8\pi G}{3\rho_m}} \left(\sqrt{1 + \frac{\Omega_\gamma}{\Omega_m}} - \sqrt{\frac{\Omega_\gamma}{\Omega_m}}\right),$$

meaning that $a_{\mathrm{eq}} = \Omega_\gamma/\Omega_m = \rho_\gamma/\rho_c$ is the scale factor at which the densities of radiation and matter are equal, marking a crossover between the two scaling regimes. The convergence of this integral also indicates that this model has a past horizon. For the future horizon, we compute

$$\int_{t_0}^{\infty} \mathrm{d}\eta = \frac{1}{H_0 \Omega_m^{1/2}} \int_1^{\infty} \mathrm{d}a \, \frac{1}{\sqrt{a + \Omega_\gamma/\Omega_m}} \to \infty,$$

which is clearly divergent, implying that there is no future horizon.

24. Consider a generalised Minkowski space:

$$ds^2 = \eta_{ab} d\xi^a d\xi^b = -d\xi_0^2 + \sum_{i=1}^{n} d\xi_i^2.$$

de-Sitter spacetime is the maximally symmetric sub-manifold described by the constraint that

$$-\xi_0^2 + \sum_{i=1}^{n} \xi_i^2 = \alpha^2. \tag{20}$$

The Riemann tensor for such a space is given by

$$R_{\text{abcd}} = \frac{1}{\alpha^2}(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

By considering Einstein's field equations in a vacuum with a non-zero cosmological constant Λ, find a relationship between α and Λ in n dimensions.

Solution: Einstein's field equations in a vacuum with non-zero cosmological constant Λ read

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = -\Lambda g_{ab}.$$

Now, it follows from the expression for R_{abcd} that the Ricci tensor is given by

$$R_{bd} = g^{ac}R_{abcd} = \frac{1}{\alpha^2}(n g_{bd} - g^c{}_d g_{bc}) = \frac{n-1}{\alpha^2} g_{bd},$$

while the Ricci scalar is

$$R = g^{bd}R_{bd} = \frac{n(n-1)}{\alpha^2}.$$

Substituting these results into the field equation, we find that

$$\alpha^2 = \frac{(n-1)(n-2)}{2\Lambda}.$$

Consider the parametrisation

$$\xi_0 = \sqrt{\alpha^2 - r^2}\sinh\left(\frac{t}{\alpha}\right), \quad \xi_1 = \sqrt{\alpha^2 - r^2}\cosh\left(\frac{t}{\alpha}\right),$$
$$\xi_2 = r\cos\theta, \quad \xi_3 = r\sin\theta\cos\phi, \quad \xi_4 = r\sin\theta\sin\phi.$$

Show that this satisfies the constraint (20), and that this gives rise to the interval

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

for $n = 4$. An *event horizon* is a hypersurface in spacetime that can only be crossed in one direction. Does this de-Sitter spacetime have such a horizon? Illustrate your answer using light-cone diagrams, distinguishing between the cases of $\Lambda > 0$ and $\Lambda < 0$.

Solution: Consider

$$
\begin{aligned}
&-\xi_0^2+\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2\\
&=-(\alpha^2-r^2)\sinh^2\left(\frac{t}{\alpha}\right)+(\alpha^2-r^2)\cosh^2\left(\frac{t}{\alpha}\right)+r^2(\cos^2\theta+\sin^2\theta(\cos^2\phi+\sin^2\phi))\\
&=(\alpha^2-r^2)+r^2=\alpha^2,
\end{aligned}
$$

meaning that it indeed satisfies the given constraint. Then, using

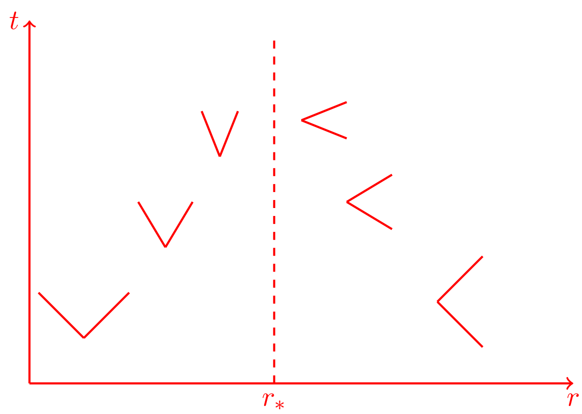
$$
\begin{aligned}
\mathrm{d}\xi_0 &= \frac{1}{\alpha}\sqrt{\alpha^2-r^2}\cosh\left(\frac{t}{\alpha}\right)\mathrm{d}t+\frac{r}{\sqrt{\alpha^2-r^2}}\sinh\left(\frac{t}{\alpha}\right)\mathrm{d}r,\\
\mathrm{d}\xi_1 &= \frac{1}{\alpha}\sqrt{\alpha^2-r^2}\sinh\left(\frac{t}{\alpha}\right)\mathrm{d}t+\frac{r}{\sqrt{\alpha^2-r^2}}\cosh\left(\frac{t}{\alpha}\right)\mathrm{d}r,\\
\mathrm{d}\xi_2 &= \cos\theta\mathrm{d}r-r\sin\theta\mathrm{d}\theta,\\
\mathrm{d}\xi_3 &= \sin\theta\cos\phi\mathrm{d}r+r\cos\theta\cos\phi\mathrm{d}\theta-r\sin\theta\sin\phi\mathrm{d}\phi,\\
\mathrm{d}\xi_4 &= \sin\theta\sin\phi\mathrm{d}r+r\cos\theta\sin\phi\mathrm{d}\theta+r\sin\theta\cos\phi\mathrm{d}\phi,
\end{aligned}
$$

we have that

$$
\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}\xi_0^2+\mathrm{d}\xi_1^2+\mathrm{d}\xi_2^2+\mathrm{d}\xi_3^2+\mathrm{d}\xi_4^2\\
&=-\left(1-\frac{r^2}{\alpha^2}\right)\mathrm{d}t^2+\left(1-\frac{r^2}{\alpha^2}\right)^{-1}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2).
\end{aligned}
$$

The desired expression follows by using the relationship between α and Λ for n=4.

This metric has an event horizon at $r=\sqrt{3/\Lambda}$ for $\Lambda>0$, and no event horizon for $\Lambda<0$. This is because g_{00} and g_{rr} reverse their character at this value of r, corresponding to a $-\pi/2$ rotation of light cones within the space. This means that trajectories at $r>\sqrt{3/\Lambda}$ are causally bounded to remain within the cosmological event horizon. This means that trajectories at $r>r_*$ are causally bounded to remain within the cosmological event horizon. This is most easily illustrated with an appropriate spacetime diagram showing 'time' orientated light cones for $r<r_*$, and 'radially' orientated light cones for $r>r_*$.



25. The angle $\Delta\theta$ subtended by some object of size d is given by $\Delta\theta = d/D_A$, where D_A is the *angular distance* of the object. By considering the components of the FRW metric (16), find an expression for D_A in terms of the radial coordinate distance to the object and the redshift z.

Solution: For a static object at a given radius r from the observer, the FRW metric (16) gives $\mathrm{d}s = ar\mathrm{d}\theta$, meaning that

$$D_A = \frac{d}{\Delta\theta} = ar = \frac{r}{1+z}.$$

The *luminosity distance* D_L is defined such that the flux F of a body of luminosity L are related by $F = L/(4\pi D_L)^2$, and is related to the angular distance by $D_L = (1+z)^2 D_A$. Show that the flux per unit area B of a body of size d and luminosity L is given by

$$B = \frac{L}{\pi^2 d^2}\frac{1}{(1+z)^4}.$$

Why is it so hard to observe old stars and galaxies?

Solution: The angular size of the object observed in the sky is given by $\Delta\theta = d/D_A$, as before. If B is the flux per unit area,

$$B = \frac{F}{\Omega} = \frac{L}{4\pi D_L^2}\frac{1}{\Omega},$$

where Ω is the solid angle subtended by the object at the observer. Assuming that the galaxy is viewed perfectly head on,

$$\Omega \simeq \frac{\text{Area}}{(\text{Distance to object})^2} = \pi\left(\frac{\Delta\theta}{2}\right)^2 = \pi\left(\frac{d}{2D_A}\right)^2 \quad \Rightarrow \quad B = \frac{L}{\pi^2 d^2}\frac{D_A^2}{D_L^2}.$$

Using the given relationship between D_A and D_L, the desired result follows. Thus, it is very hard to observe old stars and galaxies as the measured brightness has a strong (inverse) dependence on redshift z.